Hindawi
Mathematical Problems in Engineering
Volume 2021, Article ID 2264871, 6 pages
https://doi.org/10.1155/2021/2264871

Research Article

Adjustable Piecewise Quartic Hermite Spline Curve with Parameters

Jin Xie [1] and Xiaoyan Liu [2]

[1] *School of Artificial Intelligence and Big Data, Hefei University, Hefei 230601, China*
[2] *Department of Mathematics, University of La Verne, La Verne 91750, CA, USA*

Correspondence should be addressed to Jin Xie; hfuuxiejin@126.com

Received 7 October 2021; Accepted 15 November 2021; Published 28 November 2021

Academic Editor: Shaojian Qu

In this paper, the quartic Hermite parametric interpolating spline curves are formed with the quartic Hermite basis functions with parameters, the parameter selections of the spline curves are investigated, and the criteria for the curve with the shortest arc length and the smoothest curve are given. When the interpolation conditions are set, the proposed spline curves not only achieve C^1-continuity but also can realize shape control by choosing suitable parameters, which addressed the weakness of the classical cubic Hermite interpolating spline curves.

1. Introduction

In CAGD&CG, it is always an important research topic to adjust and control the shape of fitting curves. Different kinds of spline curves with parameters were constructed to control the shape of interpolating curves. For example, the Bézier-type and B-spline curves with parameters are discussed in the articles [1–4].

The classical cubic Hermite interpolation spline curves have been widely used in practical engineering problems in [5–7]. Nevertheless, when interpolation conditions are given, the shape of the cubic Hermite interpolation spline curves could not be changed. That is to say, we need to change interpolation conditions to modify the shape of the spline curves. However, when the interpolation conditions are taken from the real problems, this approach is not desirable. To overcome the limitations of the cubic Hermite interpolation spline curves in shape adaptability, the construction of Hermite interpolation spline with parameters has attracted the attention of many scholars. For example, the rational cubic Hermite interpolation spline with parameters is developed in [8–13], the quartic Hermite interpolation spline with parameters is developed in [14], and the cubic triangular Hermite interpolation spline with parameters is developed in [15]. Those articles proposed several interpolation spline functions with parameters, which have similar properties to the classical cubic Hermite interpolation, and they can push the curve to the designated area by modifying the parameters. However, the subject of how to select the parameters to obtain "good" fitting curves was not discussed in the abovementioned articles.

In this paper, a class of polynomial Hermite interpolation spline with two parameters is constructed. The shape of the spline curve could be adjusted by amending the parameters when the interpolation conditions are satisfied. One could choose the appropriate parameters to fulfill the given criterion. More specifically, we inspected the techniques to determine the parameters such that the quartic Hermite spline curve has the shortest arc length or has the least curve energy value and makes the quartic Hermite spline curve the smoothest or achieves the minimal sum of the arc length and the curve energy value.

2. Basic Concepts of Cubic Hermite Interpolation Spline

Definition 1. For $0 \leq t \leq 1$, the following four functions

$$\begin{cases} a_0(t) = 1 - 3t^2 + 2t^3, \\ a_1(t) = 3t^2 - 2t^3, \\ b_0(t) = t - 2t^2 + t^3, \\ b_1(t) = -t^2 + t^3, \end{cases} \quad (1)$$

are called the cubic Hermite basis functions, and its properties are as follows:

(1) End-points:

$$\begin{pmatrix} a_0(0) & a_1(0) & b_0(0) & b_1(0) \\ a_0(1) & a_1(1) & b_0(1) & b_1(1) \\ a_0'(0) & a_1'(0) & b_0'(0) & b_1'(0) \\ a_0'(1) & a_1'(1) & b_0'(1) & b_1'(1) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (2)$$

(2) Quasisymmetry:
$a_0(t) = a_1(1-t), b_0(t) = -b_1(1-t)$.

The function's graphs are shown in Figure 1. From left to right are the graphs of the basis functions $a_0(x)$, $a_1(x)$, $b_0(x)$, and $b_1(x)$.

Definition 2. Given data points $p_i\,(i = 0, 1, \ldots, n)$ and corresponding tangent vector p_i', for $0 \le t \le 1$, piecewise cubic Hermite interpolant curves $\mathbf{r}_i(t)$ are defined as follows:

$$\begin{aligned} \mathbf{r}_i(t) = a_0(t)p_i + a_1(t)p_{i+1} + b_0(t)p_i' \\ + b_1(t)p_{i+1}', \quad i = 0, 1, \ldots, n-1, \end{aligned} \quad (3)$$

where $a_0(t), a_1(t), b_0(t)$, and $b_1(t)$ are given in (1).

By straightforward calculation, we have

$$\begin{aligned} \mathbf{r}_i(0) &= p_i, \\ \mathbf{r}_i(1) &= p_{i+1}, \\ \mathbf{r}_i'(0) &= p_i', \\ \mathbf{r}_i'(1) &= p_{i+1}'. \end{aligned} \quad (4)$$

3. Quartic Interpolation Spline with Parameters

From the last section, we can see that for given interpolation and derivative values at designated interpolation points, the shape of the classical cubic Hermite interpolation curve is set. If you want to change the shape of the curve, you need a different set of data, which is not allowed in some situations. To modify the curve shape without changing the interpolation conditions we will construct a basis of quartic polynomial functions with parameters, when the parameters take different values, the shape of the corresponding new interpolating curves will be changed, which will be very convenient for interactive designing. If the appropriate parameter values are chosen, it can also improve the approximation effect of the interpolation curve.

Definition 3. For any given parameters α_i, β_i and $0 \le t \le 1$, the following four functions

$$\begin{cases} ea_0(t) = 1 + (\alpha_i - 3)t^2 + 2(1 - \alpha_i)t^3 + \alpha_i t^4, \\ ea_1(t) = (3 - \alpha_i)t^2 + 2(\alpha_i - 1)t^3 - \alpha_i t^4, \\ eb_0(t) = t + (\beta_i - 2)t^2 + (1 - 2\beta_i)t^3 + \beta_i t^4, \\ eb_1(t) = -(\beta_i + 1)t^2 + (2\beta_i + 1)t^3 - \beta_i t^4, \end{cases} \quad (5)$$

are called quartic Hermite basis functions with parameters.

When $\alpha_i = \beta_i = 0$, equation (5) degenerates to equation (1), which are the cubic Hermite basis functions. When α_i, β_i take different values, the shape of the basis function will also be altered. Figure 2 shows the basis function of $\alpha_i = \beta_i = 0$ (solid line), $\alpha_i = \beta_i = 1$ (dashed line), and $\alpha_i = \beta_i = -2$ (dot-dash line). From left to right are the graphs of the bases $ea_0(x)$, $eb_1(x)$, $eb_0(x)$, and $eb_1(x)$, respectively.

Definition 4. Let $a = x_0 < x_1 < \cdots < x_n = b$ be a subdivision of the interval [a, b], and $h_i = x_{i+1} - x_i$, $t = ((x - x_i)/h_i)$, then the following functions on the interval $[x_i, x_{i+1}]$

$$\begin{aligned} \mathbf{er}_i(x)\big|_{[x_i, x_{i+1}]} = ea_0(t)p_i + ea_1(t)p_{i+1} + eb_0(t)h_i p_i' \\ + eb_1(t)h_i p_{i+1}', \quad i = 0, 1, \ldots, n-1, \end{aligned} \quad (6)$$

are called quartic Hermite interpolation curves with parameters, where $ea_0(t)$, $ea_1(t)$, $eb_0(t)$, and $eb_1(t)$ are quartic Hermite basis functions with parameters given in (3).

When all parameters are $\alpha_i = \beta_i = 0$, equation (6) degenerates to equation (3).

4. Properties of the Quartic Hermite Interpolation Spline

4.1. Continuity. The spline (6) is the piecewise polynomial curves. We need to show the continuity of the curves. For $i = 1, 2, \ldots, n-1$, we have

$$\begin{aligned} \mathbf{er}_i(x_i) &= p_i, \\ \mathbf{er}_i(x_{i+1}) &= p_{i+1}, \\ \mathbf{er}_i(x_i) &= p_i', \\ \mathbf{er}_i(x_{i+1}) &= p_{i+1}'. \end{aligned} \quad (7)$$

Thus, we obtain

$$\mathbf{er}^{(k)}(x_j) = p_j^{(k)}, \quad k = 0, 1, j = i, i+1, i = 0, 1, \ldots, n-1. \quad (8)$$

That implies the spline (6) is C^1-continuous. Also, the tangent line of curves $\mathbf{er}(x)$ at the point $\mathbf{er}(x_i)$ is parallel to the tangent vector p_i' (for any α_i, β_i).

4.2. Total and Local Adjustment. By rewriting (6), for $x \in [x_i, x_{i+1}]$, we have

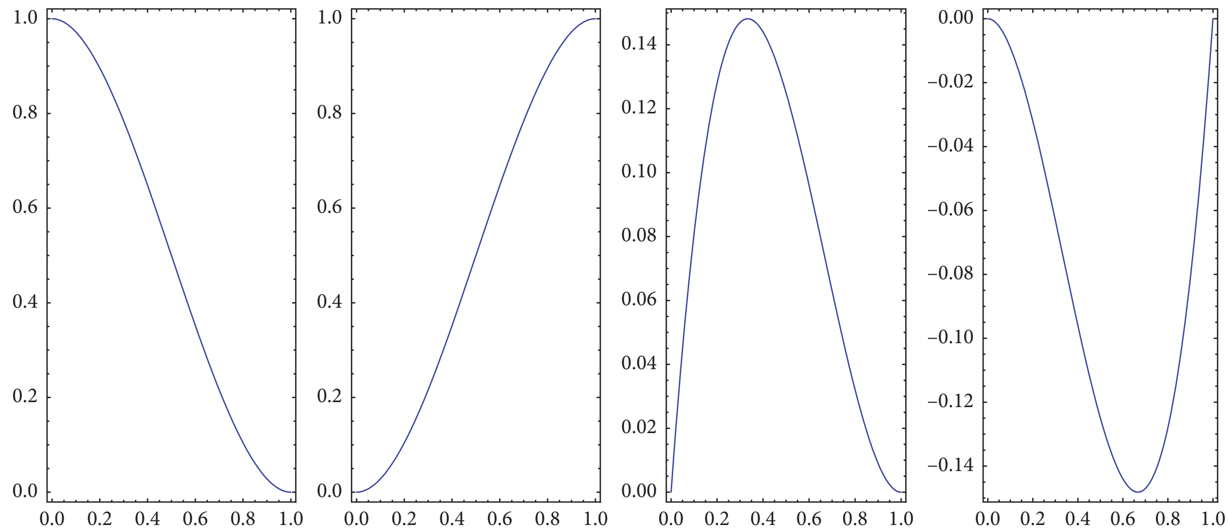

Figure 1: Hermite basis function.

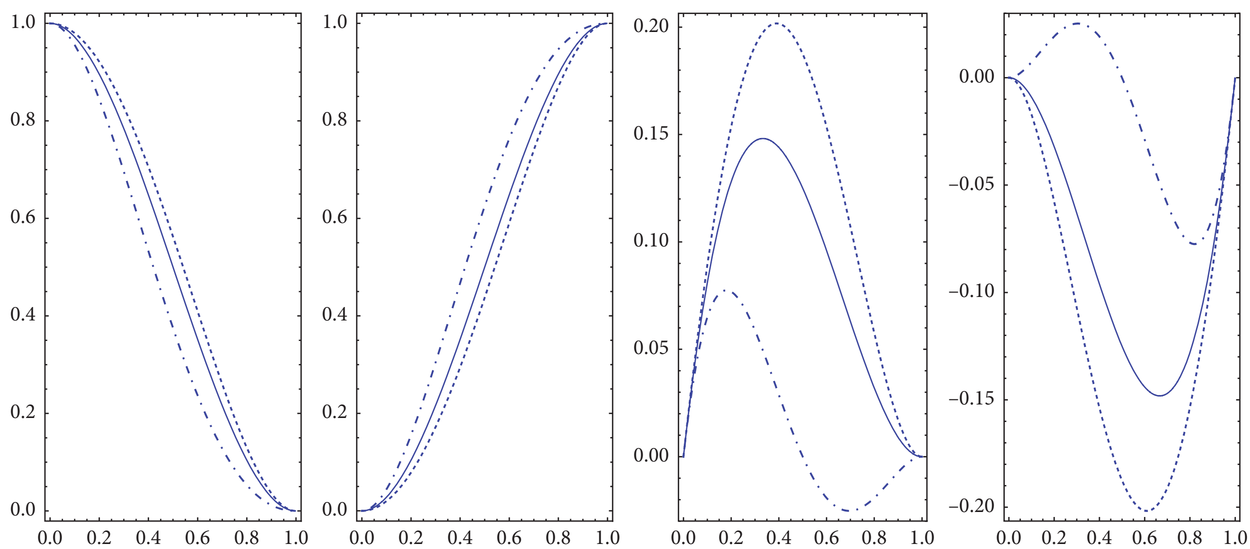

Figure 2: Quartic Hermite basis functions with different parameters.

$$\mathbf{er}(x_i)\big|_{[x_i,x_{i+1}]} = ea_0(t,\alpha_i)p_i + ea_1(t,\alpha_i)p_{i+1} + eb_0(t,\beta_i)h_i p_i' + eb_1(t,\beta_i)h_i p_{i+1}', \quad i = 0,1,\ldots,n-1. \tag{9}$$

Obviously, the parameters α_i, β_i only affect the i-th curve segments $\mathbf{er}_i(x)$ without altering the other curve segments. So, we can adjust the shape of the interpolation curves by changing parameters α_i, β_i locally.

Example 1. Given data points $p_0 = (0,2)$, $p_1 = (\sqrt{2},\sqrt{2})$, $p_2 = (2,0)$, $p_3 = (\sqrt{2},-\sqrt{2})$, $p_4 = (0,-2)$, $p_5 = (-\sqrt{2},-\sqrt{2})$, $p_6 = (-2,0)$, and $p_7 = (-\sqrt{2},\sqrt{2})$ and the corresponding tangent vectors $p_0' = (\sqrt{2},0)$, $p_1' = (1,-1)$, $p_2' = (0,-\sqrt{2})$, $p_3' = (-1,-1)$, $p_4' = (-\sqrt{2},0)$, $p_5' = (-1,1)$, $p_6' = (0,\sqrt{2})$, and $p_7' = (1,1)$, Figure 3 shows the total and local adjustable interpolation curve. The solid line in Figure 3 is the classical cubic interpolation spline curve, which is equivalent to the parameter $\alpha_i = \beta_i = 0$ in equation (6). In Figure 3(a), the parameter values of the line of dashes are $\alpha_i = \beta_i = 1$ $(i = 0,1,2,\ldots,7)$ and the parameter values of dot-dash lines are $\alpha_i = \beta_i = -2$ $(i = 0,1,2,\ldots,7)$. As we could see, the shape of the curve is adjusted as a whole. In Figure 3(b), the parameters are equal to 0, except for $\alpha_0 = 0, \beta_0 = 2, \alpha_7 = 2, \beta_7 = 0$, and the curve is adjusted locally in the first and 8-th segments.

5. Parameter Selection for Constraint Conditions

As mentioned in Section 4, the shape of the quartic Hermite interpolation spline curves can be changed by selecting suitable parameters. Theoretically, when the data points, interpolation values, and their tangent vectors remain unchanged, the shape of the interpolation curve can be adjusted arbitrarily by changing the values of parameters. However,

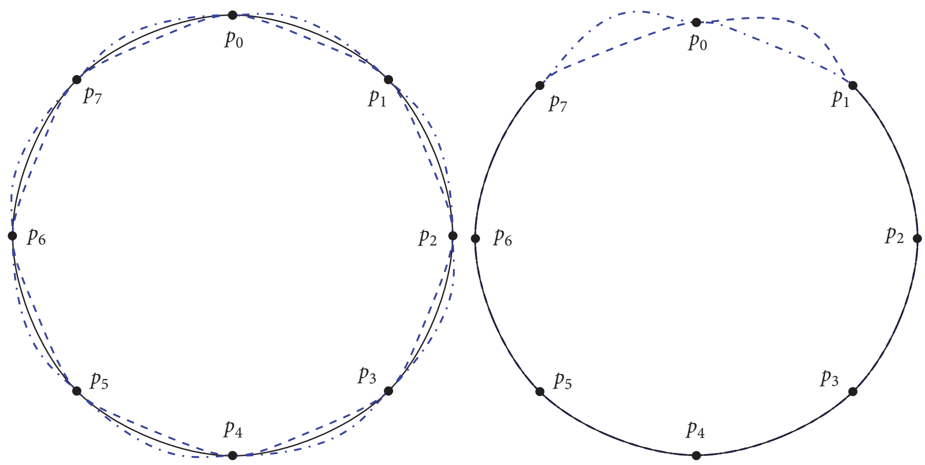

Figure 3: Total and local adjustment. (a) Total adjustment. (b) Local adjustment.

in many practical engineering problems, one often needs to make the interpolation curve meet the corresponding requirements consistent with the certain specifications. At this time, it is necessary to determine the values of parameters according to the given criteria. Three criteria for determining the values of parameters are examined as follows.

5.1. Criteria I: The Arc Length of the Curve Is the Shortest. Generally, the arc length of the parametric curve $\mathbf{r}(x)\,(a \le x \le b)$ can be expressed $L = \int_a^b |r'(x)| dx$. Therefore, for given the data points $p_i\,(i = 0, 1, \ldots, n)$ and corresponding tangent vectors $p_i'(i = 0, 1, \ldots, n)$, the optimal values of parameters α_i and β_i should be determined so that the quartic Hermite spline curve has the shortest arc length and then the optimization model can be established as follows:

$$\begin{cases} \textbf{min} & L(\alpha_i, \beta_i) = \displaystyle\sum_{i=0}^{n-1} \int_0^1 \left(r_i'(t)\right)^2 dt, \\ \text{s.t.} & \alpha_i, \beta_i \in R. \end{cases} \tag{10}$$

To find the value of the optimal parameters α_i and β_i, a system of equations relating to the partial derivatives of equations (10) can be found. We have the following equations:

$$\begin{cases} \dfrac{\partial L(\alpha_i, \beta_i)}{\partial \alpha_i} = 0, \\ & i = 0, 1, \ldots, n-1. \\ \dfrac{\partial L(\alpha_i, \beta_i)}{\partial \beta_i} = 0, \end{cases} \tag{11}$$

The interpolation curve with the shortest arc length can be obtained by solving for the optimal parameter value from the above equations.

Example 2. For the given data points $p_i\,(i = 0, 1, \ldots, n)$ and corresponding tangent vectors $p_i'(i = 0, 1, \ldots, n)$ shown in example 1, we can obtain the parameter values $\alpha_i = 0, \beta_i = -1.7\,(i = 0, 1, \ldots, 7)$ according to equations (11). The acquired quartic Hermite spline curve has the shortest arc length, as shown in Figure 4.

5.2. Criteria II: The Curve Is the Smoothest. Generally, the smoothness of an interpolation curve $r(x)\,(a \le x \le b)$ can be approximated by its energy value $E = \int_a^b (r''(x))^2 dx$, and the smaller the energy value is, the smoother the curve is. Therefore, for the given data points $p_i\,(i = 0, 1, \ldots, n)$ and corresponding tangent vectors p_i', the optimal parameters are determined to minimize the curve energy value and make the quartic Hermite spline curve the smoothest. Then, the following optimization model can be obtained:

$$\begin{cases} \textbf{min} & E(\alpha_i, \beta_i) = \displaystyle\sum_{i=0}^{n-1} \int_0^1 \left(r_i''(t)\right)^2 dt, \\ \text{s.t.} & \alpha_i, \beta_i \in R. \end{cases} \tag{12}$$

To find the optimal parameters α_i and β_i, a system of equations with the partial derivatives of the parameters α_i and β_i can be solved. The following equations are given:

$$\begin{cases} \dfrac{\partial E(\alpha_i, \beta_i)}{\partial \alpha_i} = 0, \\ & i = 0, 1, \ldots, n-1. \\ \dfrac{\partial E(\alpha_i, \beta_i)}{\partial \beta_i} = 0, \end{cases} \tag{13}$$

By solving for the optimal parameter values from the above equations, the interpolation curve with the minimum energy can be obtained.

Example 3. For the given data points $p_i\,(i = 0, 1, \cdots, n)$ and corresponding tangent vectors p_i' shown in example 1, we can attain the parameter values $\alpha_i = 0, \beta_i = 0\ (i = 0, 1, \ldots, 7)$ according to equation (13). That is to say, the classical cubic Hermite spline curve is the smoothest in this case, as the black curve shown in Figure 3(a).

5.3. Criteria III: The Shorter Arc Length and the Smoother Curve Are Both Considered. For the given data points and corresponding tangent vectors, there is an optimization

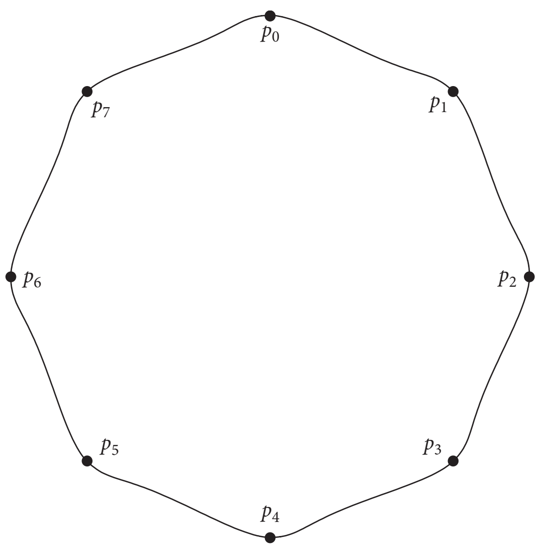

Figure 4: A closed plane curve with the shortest arc length.

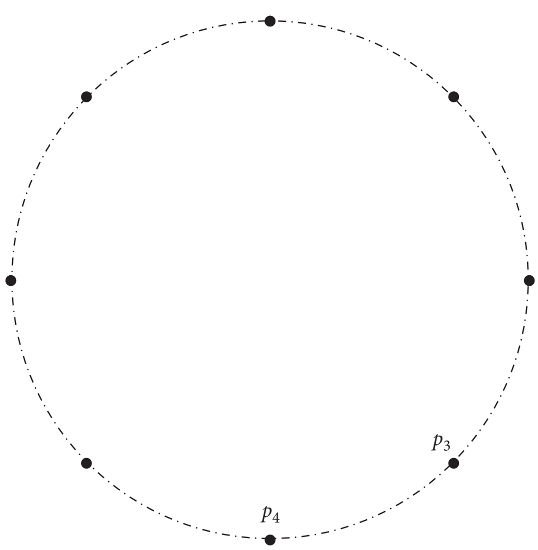

Figure 5: The plane closed curve with the least sum of the arc length and energy.

model to make the sum of the arc length and the energy of the quartic Hermite spline curve to be the smallest as follows:

$$\begin{cases} \min & F(\alpha_i, \beta_i) = L(\alpha_i, \beta_i) + E(\alpha_i, \beta_i), \\ \text{s.t.} & \alpha_i, \beta_i \in R. \end{cases} \tag{14}$$

The partial derivatives of the above equations concerning parameters α_i and β_i are obtained as follows:

$$\begin{cases} \dfrac{\partial F(\alpha_i, \beta_i)}{\partial \alpha_i} = 0, \\ & i = 0, 1, \ldots, n-1. \\ \dfrac{\partial F(\alpha_i, \beta_i)}{\partial \beta_i} = 0. \end{cases} \tag{15}$$

Example 4. For the given data points $p_i\,(i = 0, 1, \ldots, 7)$ and corresponding tangent vector p_i' given in example 1, we can obtain the parameter values $\alpha_i = 0, \beta_i = -0.0407$ $(i = 0, 1, \ldots, 7)$ according to equation (15). Thus, the sum of the arc length and the energy of the quartic Hermite interpolation spline curve is the smallest, as shown in Figure 5.

6. Conclusion

In this paper, we construct a set of adjustable quartic Hermite interpolation spline functions with two parameters, which inherits the main characteristics of the classical cubic Hermite interpolation spline functions and overcomes its limitation. When the data points and their tangent vectors are given, the shape of the classical cubic Hermite spline curve cannot be changed, which is a weakness. The shape of our proposed quartic Hermite interpolation spline curve can be adjusted by adapting the parameters, which is valuable when certain requirements of the interpolation effect in practical engineering designs need to be met. In addition, the quartic Hermite parameter spline still adopts the piecewise polynomial form, which not only has a relatively simple expression but also stays in line with the standard Bézier curve, B-spline curve, and other polynomial parameter curves in CAD/CAM system.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

The work was funded by the Key Project Foundation of Scientific Research, Education Department of Anhui Province, under Grant No. KJ2018A0555.

References

[1] S. Maqsood, M. Abbas, and G. Hu, “A novel generalization of trigonometric bézier curve and surface with shape parameters and its applications,” *Mathematical Problems in Engineering*, vol. 2020, Article ID 4036434, 25 pages, 2020.

[2] X. Qin, G. Hu, N. Zhang, X. Shen, and Y. Yang, “A novel extension to the polynomial basis functions describing Bezier curves and surfaces of degree n with multiple shape parameters,” *Applied Mathematics and Computation*, vol. 223, no. 3, pp. 1–16, 2013.

[3] J. Cao and G.-z. Wang, “Non-uniform B-spline curveswith multiple shape parameters,” *Journal of Zhejiang University - Science C*, vol. 12, no. 10, pp. 800–808, 2011.

[4] X. Han, “Normalized B-basis of the space of trigonometric polynomials and curve design,” *Applied Mathematics and Computation*, vol. 251, pp. 336–348, 2015.

[5] R. A. Lorentz, “Multivariate Hermite interpolation by algebraic polynomials: a survey,” *Journal of Computational and Applied Mathematics*, vol. 122, no. 2, pp. 167–201, 2000.

[6] A. Gfrerrer and O. Röschel, “Blended hermite interpolants,” *Computer Aided Geometric Design*, vol. 18, no. 9, pp. 865–873, 2001.

[7] J.-H. Yong and F. Cheng, "Geometric Hermite curves with minimum strain energy," *Computer Aided Geometric Design*, vol. 21, no. 3, pp. 281–301, 2004.

[8] Q. Duan, L. Wang, and E. H. Twizell, "A new C2 rational interpolation based on function values and constrained control of the interpolant curves," *Applied Mathematics and Computation*, vol. 161, no. 1, pp. 311–322, 2005.

[9] Q. Duan, H. L. Zhang, and X. Lai, "Constrained rational cubic spline and its application," *Journal of Computational Mathematics*, vol. 19, no. 2, pp. 143–150, 2001.

[10] F. X. Bao, Q. H. Sun, and J. X. Pan, "Point control of rational interpolating curves using parameters," *Mathematical and Computer Modeling*, vol. 52, no. 1, pp. 143–151, 2010.

[11] J. Xie, J. Q. Tan, and S. F. Li, "Rational cubic Hermite interpolating spline and its approximation properties," *Chinese Journal of Engineering Mathematics*, vol. 28, no. 3, pp. 385–392, 2011, (in Chinese).

[12] J. Xie, J. Q. Tan, and Z. Liu, "A class of rational cubic trigonometric Hermite interpolating splines with parameters," *Mathematica Numerica Sinica*, vol. 33, no. 2, pp. 125–132, 2011, (in Chinese).

[13] J. Xie, J. Q. Tan, and S. F. Li, "A kind of rational cubic spline with parameters and its applications," *Acta Mathematicae Applicatae Sinica*, vol. 33, no. 5, pp. 847–854, 2010, (in Chinese).

[14] J. Xie and X. Y. Liu, "The EH interpolation spline and its approximation," *Abstract and Applied Analysis*, vol. 2014, Article ID 745765, 7 pages, 2014.

[15] J. C. Li, Y. E. Zhong, and C. Xie, "Cubic trigonometric Hermite interpolating splines curves with shape parameters," *Computer Engineering and Applications*, vol. 50, no. 17, pp. 182–185, 2014, in Chinese.