MA 323 Geometric Modelling
Course Notes: Day 33
Geometry of Surfaces

David L. Finn

We have discussed several methods for creating surfaces in the past few chapter. In this chapter, we want to start talking about analyzing the shape of a surface. Recall, that the shape of a plane curve is uniquely determined by the curvature of a curve. The shape of a space curve is given by a curvature (rate of change of the unit tangent vector) and a torsion (the rate of change of the orthogonal frame). However, the shape of a surface is more complicated. It is not uniquely determined by any set of functions. We must at least give a curvature function and a boundary curve.

To determine the shape of a surface, we need two curvatures. The general method is to compute the unit normal vector of the surface, and then compute the rate of change of the normal vector in different directions. The question is what directions? Does it matter which directions?

Given a parametric surface $\mathcal{S}$ described as $\mathbf{x}(u,v)$. The normal vector is given as $\mathbf{n} = \mathbf{t_u} \times \mathbf{t_v}$, where $\mathbf{t_u}$ is the unit tangent vector in the u-direction and $\mathbf{t_v}$ is the unit tangent vector in the v-direction. The unit normal vector to the surface is well defined provided the partial derivatives $\partial \mathbf{x}/\partial u$ and $\partial \mathbf{x}/\partial v$ are linearly independent, i.e. not scalar multiplies of each other, which means the tangent plane of the surface is well-defined. To define the curvature, we need the unit normal well-defined at each point p, moreover we need a consistent choice of this unit normal. This means that the unit normal $\mathbf{n}(u,v)$ must be continuous or always point on the same side of the surface.

We compute the curvature of a surface in the following manner. Given a point p, compute the unit normal $\mathbf{n}$ vector at p. Each unit tangent $\mathbf{t}$ in the tangent plane, consider the curve $c_{\mathbf{t}}$ defined as the intersection of the surface $\mathcal{S}$ and the plane through p containing the vector $\mathbf{t}$ and $\mathbf{n}$, (see diagram below). Orient this curve so that the vector $\mathbf{t}$ is the unit tangent vector of $c_{\mathbf{t}}$ at p, then compute the second derivative of $c_{\mathbf{t}}$ with respect to arclength (the derivative of the unit tangent vector with respect to arclength). This second derivative is a multiple of $\mathbf{n}$, for $\mathbf{n}$ is a normal vector of the curve. This multiplicative factor is the curvature of the surface at p in the direction $\mathbf{t}$, $\kappa(\mathbf{t})$. We thus have a curvature for each direction in the tangent plane.

The directions in the tangent plane are equivalent to a circle. Thus, we have a function κ on the circle. This function attains a maximum value and a minimum value. The directions at which the maximum and minimum are attained are called the principle directions. A theorem of Euler states that the maximum and minimum values of the curvature occur in orthogonal directions. The curvatures in these directions are the principle curvatures, κ_1 and κ_2. Using the principle curvatures, we define the curvatures of a surface. Mathematically, there are two important curvatures, the mean curvature (the average of the principle curvatures),

$$\text{mean curvature:} \kappa_{mean} = \frac{1}{2}\left(\kappa_1 + \kappa_2\right)$$

and the Gauss curvature (the product of the principle curvatures)

$$\text{Gauss curvature:} \quad \kappa_{Gauss} = \kappa_1\kappa_2.$$

In geometric modelling, neither of these two curvatures provide the correct measure of shape. The measure of curvature that is employed in geometric modelling is either the root mean curvature

$$\text{RM Curvature:} \quad \kappa_{RMC} = \sqrt{\kappa_1^2 + \kappa_2^2},$$

the square root of the sum of the principle curvatures squared or the absolute curvature,

$$\text{absolute curvature:} \quad \kappa_{abs} = |\kappa_1| + |\kappa_2|$$

the sum of the absolute values of the principal curvatures.

We will not get into the technical details of computing the curvatures. Rather, we want to try to understand what the curvatures tell us and why we should care about curvature. We will focus on the root mean curvature. This curvature measures the deviation of the surface from a plane, as a surface with zero root mean curvature is a plane. Higher root mean curvature implies that the surface normal is changing very quickly in a small area. While small root mean curvature means that the surface normal is changing slowly. The problem with root mean curvature is that it does not tell you which direction the surface is curved. For that, one has to look at the principal curvatures.

The first use of curvature is to detect where to place control points or data points for interpolation. We need to use more control points or data points in regions of high curvature, and we need to use fewer control points to create a region of low curvature. The other uses of curvature are to analyze the interaction of the object in its surroundings, and to determine whether more analysis is needed. In fact, curvature is used in some rendering programs to aid in ray-tracing. Curvature is also used to create texture maps, especially by coloring by curvature.

For production purposes, rapidly changing surfaces normals are normally bad in terms of production. It is normally harder to produce such a surface and it normally costs more. For instance, in designing a car, a plane, or a ship, rapidly changing surface normals generally introduce a higher drag coefficient. There generally must be an advantage in producing a highly curved surface to offset the penalties. Curvature is also used to detects faults in production, as a ding or other production defect will alter the curvature. To detect curvature, a standard technique is light interaction. The interaction of light with an object

31.1   Exercises

1. Use the files *surfcurv.mws* and *surfcurv.txt* to analyze the curvatures of some patches.