MA 323 Geometric Modelling
Course Notes: Day 37
Doo-Sabin and Catmull-Clark Surfaces

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37.1 The Catmull-Clark Subdivision Method

In the Catmull-Clark method, rather than describing the generation globally, we construct the new polyhedron locally by describing what happens near each vertex v. In the procedure, we form new points (face points) f'_j corresponding to each face that v is adjacent to. We also form new points (edge points) e'_j corresponding to each edge for which v is an endpoint. Lastly, we form a new point vertex point v'.

- Each new face point f'_j is computed by finding the centroid of the face f_j.
- Each new edge point e'_j is the average of the endpoints of the edge and the face points f' that the edge is adjacent to.
- The new vertex point v' is computed as follows,

  $$v' = \frac{1}{3}v + \frac{1}{3n}\sum_{j=1}^{n} e'_j + \frac{1}{3n}\sum_{j=1}^{n} f'_j,$$

  where n is the number of edges (and faces) that surround the vertex point v. We are averaging the average of all faces adjacent to v, the average of all edges adjacent to v, and the vertex v.

We form new faces by considering loops (sequences of vertices) starting at the new vertex point v' to an edge point e'_1 to a face point f' to another edge point e'_2 back to the vertex point v'. The edge points e'_1, e'_2 must arise from edges e_1 and e_2 that share endpoint v. The face point f' should be the centroid of a face that shares the edges e_1 and e_2. Notice, that this algorithm produces faces that are four sided. Consider the diagram below, as an illustration.

Let's consider the growth in size of the polyhedron under this procedure, letting F be the number of original faces, E be the number of original edges, and V be the number of original vertices. Further, let the size of the ith iterate be (F_i, E_i, V_i). The algorithm implies that $V_{i+1} = F_i + E_i + V_i$ since each face, edge and vertex generates a new vertex. In addition, the face that each face is four sided implied that the number of edges is twice the number of current faces, i.e. $E_{i+1} = 2F_{i+1}$, since each edge belongs to two faces. The number of current faces is then given by the Euler characteristic $F_{i+1} - E_{i+1} + V_{i+1} = \xi = F_i - E_i + V_i$ or $-F_{i+1} + F_i + E_i + V_i = F_i - E_i + V_i$ which gives $F_{i+1} = 2E_i$. This described in summary below

$$\begin{aligned} F_{i+1} &= 2E_i E_{i+1} &&= 2F_{i+1} = 4E_i \\ V_{i+1} &= F_i + E_i + V_i \end{aligned}$$

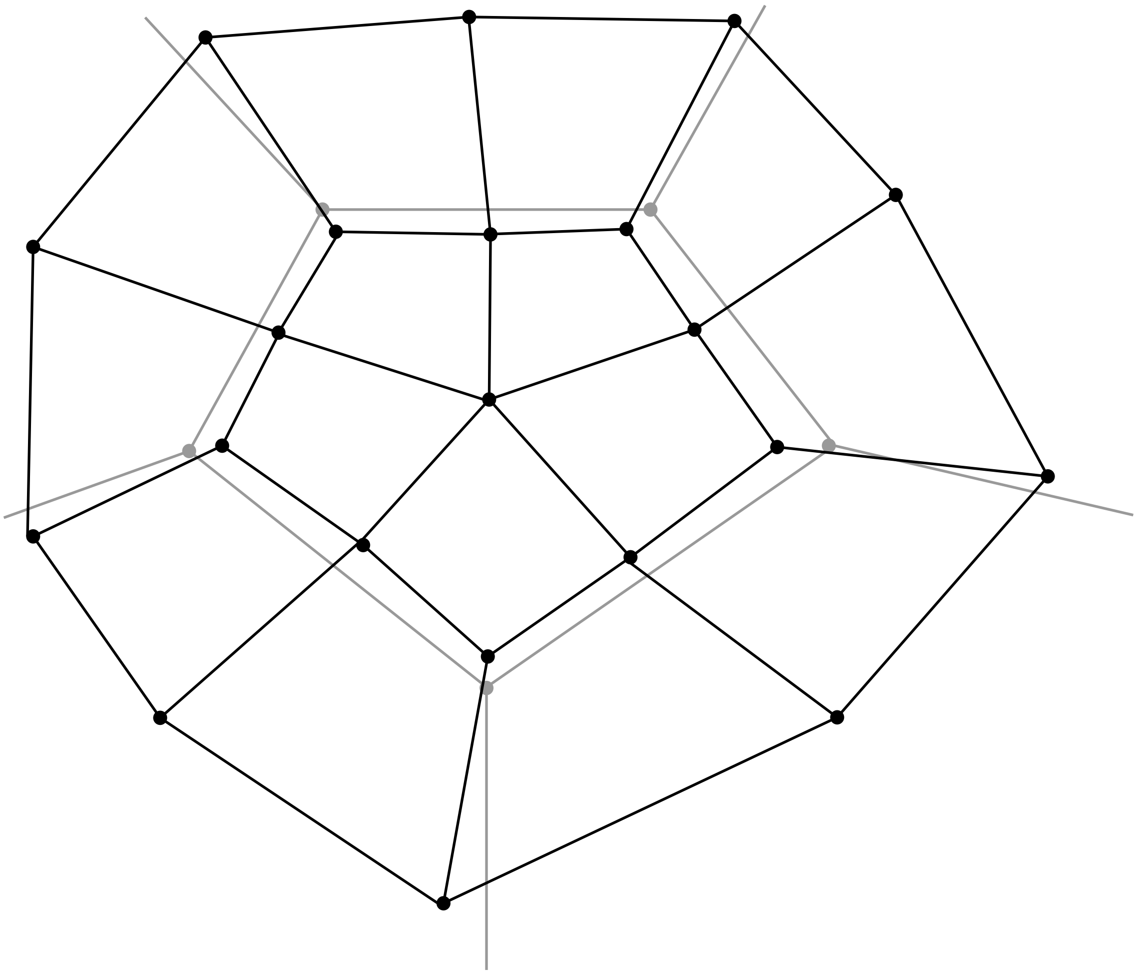

Figure 1: Catmull-Clark Method - original polyhedron in gray, new polyhedron in black

or

$$\begin{aligned} F_{i+1} &= \tfrac{1}{2}\, 4^i\, E \\ E_{i+1} &= 4^i\, E \\ V_{i+1} &= F + V + (\tfrac{1}{2}\, 4^i - 1)\, E \end{aligned}$$

which is very similar to the centroid method replacing the number of vertices with the number of faces.

Showing that the algorithm converges is a little more delicate. First, we note that like the midpoint method and the centroid method, the Catmull-Clark method satisfies the convex hull property, as each new vertex is arrived at via a convex combination of original vertices. This means that the limiting surface can not run off towards infinity. It takes a more work to show that the lengths of edges decrease, but once that is done then the convergence to a limiting surface proceeds as before.

37.2 The Doo-Sabin Subdivision Method

The Doo-Sabin method is similar yet different than the Catmull-Clark method. Starting with a polyhedron with vertices V, edges E and faces F. We define a new set of vertices, and then construct faces as in all subdivision methods.

We construct a new set of vertices for each face. Let $v_i^{(0)}$ be the vertices in a face. We construct a new set of vertices for this face based on the following iteration scheme,

$$v_i^{(1)} = \sum_{j=1}^{n} \alpha_{ij}\, v_j^{(0)}$$

where n is the number of vertices in this face, and the coefficients α_{ij} are given by

$$\alpha_{ii} = \frac{n+5}{4n}$$
$$\alpha_{ij} = \frac{3 + 2\cos(\frac{2\pi(i-j)}{n})}{4n}$$

Notice that the number of vertices constructed for each face is equal to the number of vertices in the original face. This means that the total number of points constructed is greater than the number of points in the original polyhedron, because each vertex belongs to more than one face (in fact each vertex normally belongs to at least 3 faces). It is important to notice that the sum is a convex combination as $\sum_{j=1}^{n} \alpha_{ij} = 1$.

We now construct the new faces. Each face in the original polyhedron will generate a new face. Each edge in the original polyhedron will generate a new face. And each vertex in the original polyhedron will generate a new face.

- Each original face will construct a new face based on cyclically connecting the new vertices corresponding to the face.
- Each original edge generates a face by considering the four vertices generated by the endpoints of the edge. Since each edge belongs to two faces, and based on the construction of new vertices, we get a new vertex for each face corresponding to the endpoints of the original edge. Thus, a new face is form with four edges.
- Each original vertex will generate a face by cyclically connecting the all the new vertices that the original vertex creates. The edges of this face are defined by the edges of the original polyhedra that the vertex is incident on.

We note that as this method is applied to a polyhedron, most of the faces constructed will be four sided. This is because every edge creates a four sided face and once we have a four sided face constructed it generates a four sided face. This is very similar to the centroid method. In fact, size wise the polyhedrons iteration is exactly the same. The only difference is the method for creating the vertices.

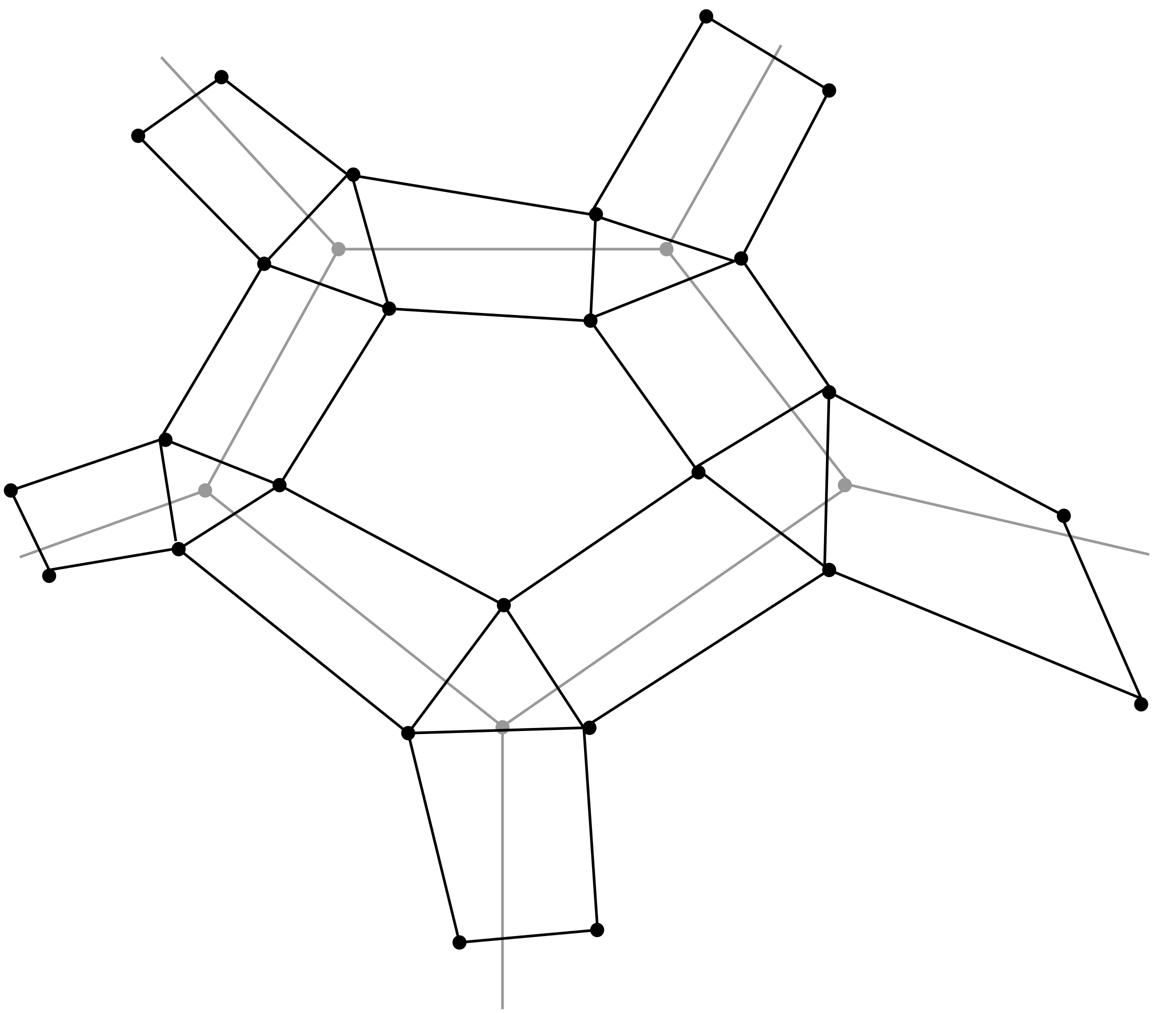

Figure 2: Doo-Sabin Method - original polyhedron in gray, new polyhedron in black

37.3 Exercises

1. Use the applets to examine the differences between the Centroid method and Doo-Sabin method. What is different qualitatively between the limiting surfaces?

2. Use the applets to examine the difference between the Catmull-Clark method and the Doo-Sabin method. What is different between the limiting surfaces?

3. In these methods you always are rounding corners. This means it is not possible to obtain a corner. How could obtain a more sharp corner? Higher curvature in the limiting surface?