

# MA 323 Geometric Modelling

## Course Notes: Day 32

### Triangular Coons Patches

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Today, we want to consider triangular Coons patches, and some other ways to construct a patches. In particular, we want to start considering different types of controls. Instead of supplying control points to construct patches by constructing the edge curves, we want to consider specifying points and corresponding normal vectors.

#### 31.1 Triangular Coons Patches

A triangular Coons patch is designed to solve the following problem.

Given three curves that form a *topological* triangle (see diagram below), construct a surface that passes through these three curves.

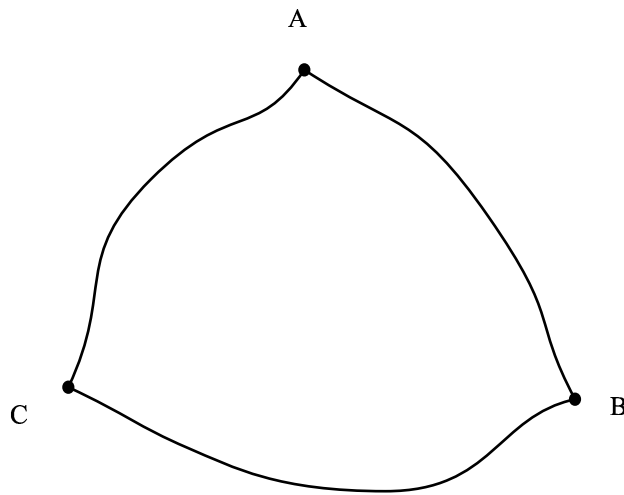


Figure 1:

To construct a triangular Coons patch, one method is to consider a degenerate rectangular Coons patch, that is treat one of the boundary curves as a point, see diagram below. A problem with this method is that it introduces a singularity in the patch structure, as the derivatives on the boundary curve that is considered as point will pose problems in joining patches.

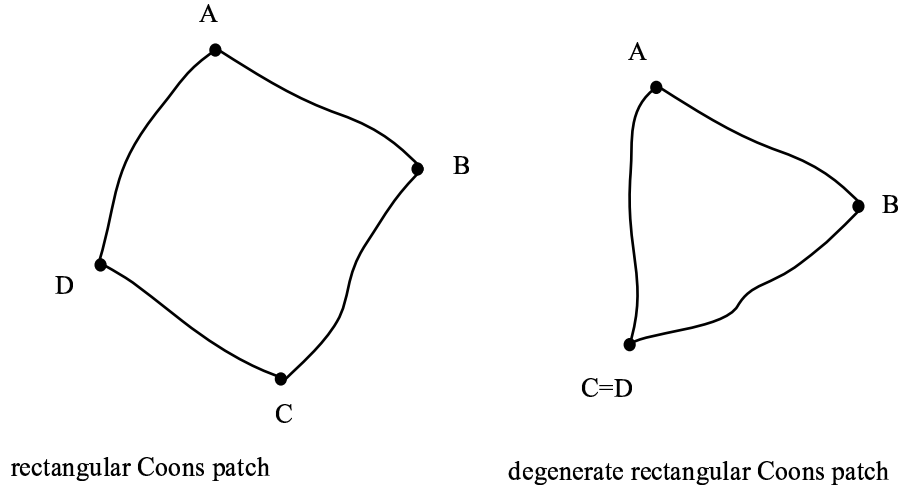


Figure 2: Degenerate rectangular Coons patch

Constructing a triangular Coons patch based on generalizing the construction of a rectangular Coons patch based on linear blending is relatively straight forward. To construct the patch, you proceed by constructing three different linearly blended patches, as illustrated in the diagrams below. Each boundary curve will generate one patch using the opposite corner, as follows using barycentric coordinates

$$\begin{aligned} X_A &= u A + (1 - u) c_{BC}(v, w) \\ X_B &= v B + (1 - v) c_{CA}(u, w) \\ X_C &= w C + (1 - w) c_{AB}(u, v) \end{aligned}$$

where  $c_{AB}$ ,  $c_{BC}$ ,  $c_{CA}$  are the boundary curves, see diagrams below. The patch is constructed by adding the three individual patches together and then subtracting off twice the triangle formed from corner points. The reason for subtracting twice the triangle comes from looking at the desired patch as

$$X = X_A + X_B + X_C - 2X_T$$

where  $X_A$ ,  $X_B$ ,  $X_C$  are the patches formed by linear blending to the corner points  $A$ ,  $B$ ,  $C$ , and  $X_T$  is the patch to be subtracted off. It must be twice a patch based on affine combinations; to obtain a point we must have the coefficients sum to one. Looking at the desired values of the  $X(u, v, 0) = c_{AB}(u, v)$ ,  $X(u, 0, w) = c_{CA}(u, w)$  and  $X(0, v, w) = c_{BC}(v, w)$ , we have

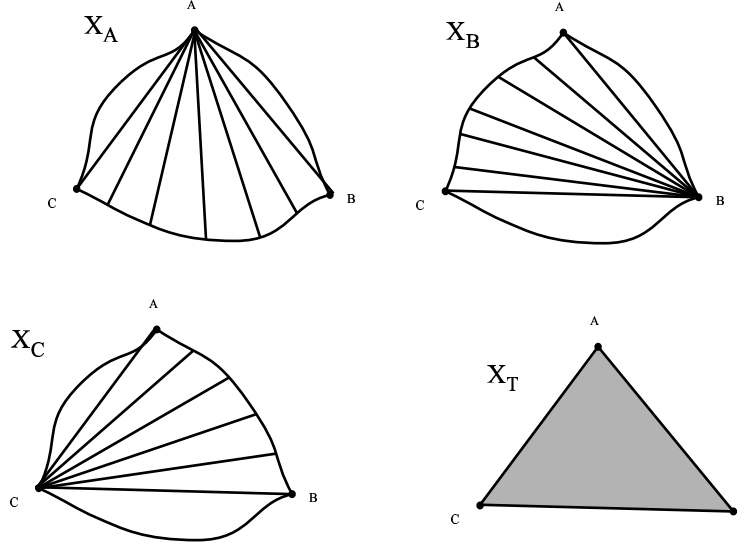
$$\begin{aligned} X(u, v, 0) &= c_{AB}(u, v) = c_{AB}(u, v) + uA + vB + uA + vB - 2X_T(u, v, 0) \\ X(u, 0, w) &= c_{CA}(u, w) = uA + wC + c_{CA}(u, w) + uA + wC - 2X_T(u, 0, w) \\ X(0, v, w) &= c_{BC}(v, w) = vB + wC + vB + wC + c_{BC}(v, w) - 2X_T(0, v, w) \end{aligned}$$

thus

$$\begin{aligned} X_T(u, v, 0) &= uA + vB \\ X_T(u, 0, w) &= uA + wC \\ X_T(0, v, w) &= vB + wC \end{aligned}$$

which are the edges of the triangle. We need for  $X_T$  a patch that interpolates the straight lines forming a triangle. A simple choice is thus the triangle  $X_T = uA + vB + wC$ , any other

surface interpolating the straight lines would work, but constructing such a surface would require more work.



Linearly blended triangular Coons patch construction  

$$\mathbf{X} = \mathbf{X}_A + \mathbf{X}_B + \mathbf{X}_C - 2\mathbf{X}_T$$

Figure 3: Construction of a linearly blended triangular Coons patch

Like joining rectangular Coons patches that are obtained from linear blending, joining triangular Coons patches using linear blending will not be smooth. To join smoothly, we need more flexibility in the blending. We need at least cubic blending, however as we saw in joining triangular Bezier patches, joining triangular patches is not as easy as joining rectangular patches.

### 31.2 A cubic method for triangular Coons patches

Constructing a cubically blended triangular Coons patch requires a little more work than for rectangular Coons patches. The setup is similar though modified slightly for triangles, see diagram below. Proceeding similar to the construction of a linearly blended triangular Coons patch, we first construct a cubic blending from each base point to the each boundary curve; written for  $A$  and  $c_{BC}$  we get

$$\begin{aligned} X_A = & u^3 A + 3(A + v \frac{1}{3} T_A B(1, 0) + w \frac{1}{3} T_A C(1, 0)) u^2 (1 - u) \\ & + 3(c_{AB}(v, w) - v \frac{1}{3} T_A B(0, 1) - w \frac{1}{3} T_A C(0, 1)) u (1 - u)^3 + c_{AB}(v, w) \end{aligned}$$

where  $T_{AB}(u, v)$  and  $T_{AC}(u, w)$  are the derivatives along the curves  $c_{AB}$  and  $c_{AC}$ . Then, we look at the Coons patch as

$$X = X_A + X_B + X_C - 2X_T$$

where  $X_T$  is a patch that passes through the three edge cubic Hermite curves given from the derivatives of the edge curves at the corner points. One way to build  $X_T$  is to cubically blend from  $A$  to the cubic curve interpolated from  $B$  and  $C$ , see diagram below. For symmetry reasons, it is useful to construction  $X_T$  from the cubic Hermite curves for each edge, and then average these patches.

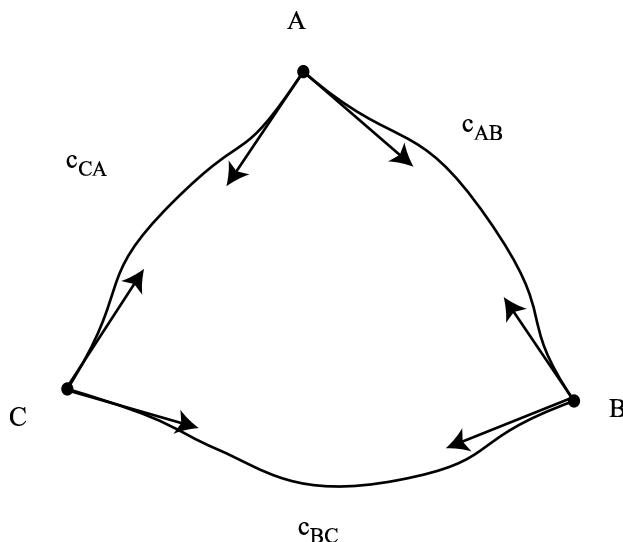


Figure 4: Construction of Cubically Blended Triangular Coons Patch

To join two triangular cubic Coons patches across a common edge, we proceed by continuing each curve smoothly through the joint is done in the same manner as joining two cubic curve at a joint point. This creates a problem if you want to join multiple triangular patches at a corner.

### 31.3 A Hermite Interpolation Problem for Surfaces

The difficulty in constructing smooth joinings of triangular patches is because of problem in constructing the vector fields used in blending across the boundary curves. An alternate construction is to look at different type of interpolation problem. This difficulty is a result of needing to define the tangent planes at each point on the boundary curve. The tangent plane at the corner points of the triangle are well-defined by the boundary curves, assuming compatibility conditions. However, the tangent planes along the boundary curves are not necessarily well-defined by the blending method along the boundary curves.

An alternate method is not to specify the boundary curves but rather to specify the tangent plane at the corner points, more specifically the normal vector to the tangent plane. This is then analogous to the construction of Hermite curve, but for a patch. The question is then to construct a surface passing through a collection of specified points and having prescribed tangent planes (normal vectors) at these points, see figure below. This problem is different than any other type of patch problem we have considered, as we have very little information now, and it is not immediate how to solve this problem with curve methods. However, this is really the correct interpolation problem that one encounters in practice. When one samples information off a surface, it is easy to get both information about a point on the surface and the normal vector to the surface at that point.

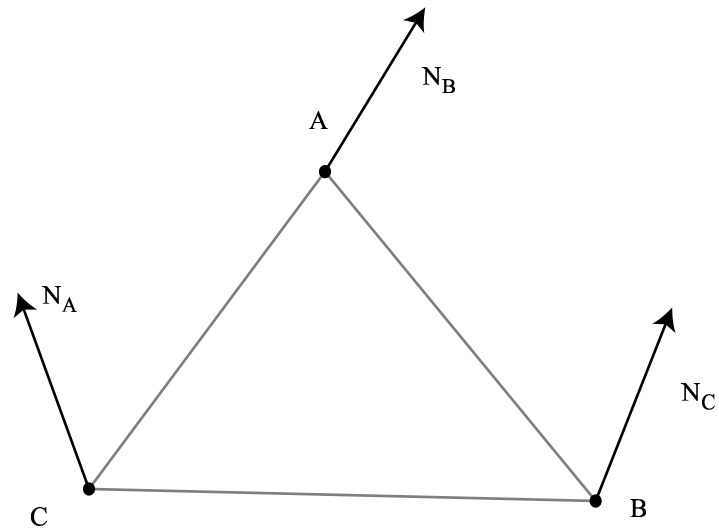


Figure 5: A Hermite type patch interpolation problem for a triangular grid

Tomorrow we will look at one method for solving this problem using a subdivision scheme. Here is a thought problem, how does one go about constructing a solution to this type of problem?