MA 323 Geometric Modelling
Course Notes: Day 07
Parabolic Arcs

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We now start considering the basic curve elements to be used throughout this course; polynomial curves and piecewise polynomial curves. These types of curves have component functions that are polynomial functions or piecewise polynomial functions. One of the reasons for the use of polynomial curves (and later polynomial surfaces) in geometric modelling is that the are easy to calculate, and numerically efficient to calculate, at least once put in the correct form. A second reason is the great availability of results on polynomials; they have mathematically studied for centuries.

An nth degree polynomial in t is an expression involving a series of powers of the variable t multiplied by a series of coefficients, that is

$$f(t) = a_0 + a_1\, t + a_2\, t^2 + \cdots a_n\, t^n.$$

The degree (or order) of the polynomial is typically the highest power with a non-zero coefficient. However, for our purposes, we use a slightly different notion of the order of a polynomial. We will typically fix the highest power of the polynomials, and thus consider a nth degree polynomial to mean the polynomial involves powers of t that are at most n, meaning that a_n can equal zero and it is still considered an nth degree polynomial. This allows us to consider a quadratic polynomial, $a_0 + a_1\, t + a_2\, t^2$ as a cubic polynomial with $a_3 = 0$. But, the real advantage is that it allows us to consider the space of polynomials of degree n as a vector space of dimension $n + 1$, and apply the techniques and computational power of linear algebra to polynomial curves.

A polynomial curve is a vector valued function whose coordinate functions are polynomials, that is for a polynomial curve of degree n in the xy-plane the x component and the y component functions are polynomials of degree n,

$$x(t) = a_0 + a_1\, t + a_2\, t^2 + \cdots + a_n\, t^n \quad \text{and} \quad y(t) = b_0 + b_1\, t + b_2\, t^2 + \cdots + b_n\, t^n$$

Exploiting vector algebra, we can represent a polynomial curve $c(t)$ of degree n as a single polynomial of degree n with vector coefficients,

$$c(t) = \mathbf{a_0} + \mathbf{a_1}\, t + \mathbf{a_2}\, t^2 + \cdots + \mathbf{a_n}\, t^n.$$

With this formulation, much of the notation is simplified. Moreover, this formulation easily allows us to apply the techniques of linear algebra to solve problems.

We will also start considering the idea of piecewise polynomial curves. These curves are obtained by joining one polynomial curve to another polynomial curve either in a continuous manner or a smooth manner, for instance

$$c(t) = \begin{cases} \mathbf{a}_0^1 + \mathbf{a}_1^1\, t + \mathbf{a}_2^1\, t^2 + \cdots + \mathbf{a}_n^1\, t^n & \text{for } t_0 \le t < t_1 \\ \mathbf{a}_0^2 + \mathbf{a}_1^2\, t + \mathbf{a}_2^2\, t^2 + \cdots + \mathbf{a}_n^2\, t^n & \text{for } t_1 \le t < t_2 \\ \vdots & \vdots \\ \mathbf{a}_0^m + \mathbf{a}_1^m\, t + \mathbf{a}_2^m\, t^2 + \cdots + \mathbf{a}_n^m\, t^n & \text{for } t_{m-1} \le t \le t_m. \end{cases}$$

The curve $c(t)$ defined above uses the same degree of polynomials for each segment. This is not necessary, but is a notational convenience as we can increase the degree without changing the curve since we will allow $\mathbf{a}_n^k = 0$.

For piecewise polynomial curves, it will be important to consider the order of smoothness of the curve. The order of smoothness of the curve is based on the number of derivatives that are defined at the point where the curves are joined. When considering the order of smoothness of the curve, it is important to distinguish whether or not we are considering parametric smoothness or geometric smoothness. Parametric smoothness means the derivatives of the polynomial agree in the parameter t. Geometric smoothness means that with respect to the arc-length parameter the curve is parametrically smooth. For geometric modelling, geometric smoothness is important because the arc-length parameter is the natural parameter in geometry, and thus the natural parameter in designing objects. Geometric smoothness adds more freedom to the designer, with the extra cost of algorithm complexity as the arc-length parameter is not the natural parameter for computations with polynomial curves.

The remainder of today's notes concerns quadratic polynomial curves (parabolic arcs). We will solve all the motivating problems with parabolic arcs, in detail. Tomorrow, we will consider cubic curves in some generality looking at an interpolation problem. In considering, cubic curves, we will also look at the problem of degree elevation and degree reduction, and the construction of piecewise cubic curves. After cubic curves, we will consider how the solution of the motivating problems with parabolic curves and cubic curves generalizes to higher-order polynomials curves.

7.1 Parabolic Arcs and Piecewise Parabolic Arcs

We begin our discussion of polynomial curves and piecewise polynomial curves with parabolic arcs. Parabolic arcs are standardly represented as nondegenerate quadratic curves, that is second order polynomial curves. The nondegeneracy condition ensures that the quadratic curve is not a straight line. Unlike other polynomial curves, parabolas have a well-defined geometric definition. A parabola is the set of points P in a plane that are equally distance from a given line L (called the directrix) and a given point F (called the focus), see figure below. This is definition from Euclidean geometry. Parabolas also naturally arise when considering conic sections, the intersection of plane with a cone, which we encounter later in these notes when we consider projective geometry and rational quadratic curves.

It is relatively easily application of basic ideas in analytic geometry to show that a parabola is described by the equation $y = ax^2 + c$. To explain the derivation of this formula from the geometric definition. Let Q be the intersection of the directrix and the line perpendicular to the directrix through F and let V (the vertex of the parabola) be the point on the parabola on the line segment QF, as in the diagram above and below. The quantity x is then the distance between a point X on the directrix and the point Q, and y is the distance on a perpendicular to the directrix to the parabola. From the geometric definition, c must

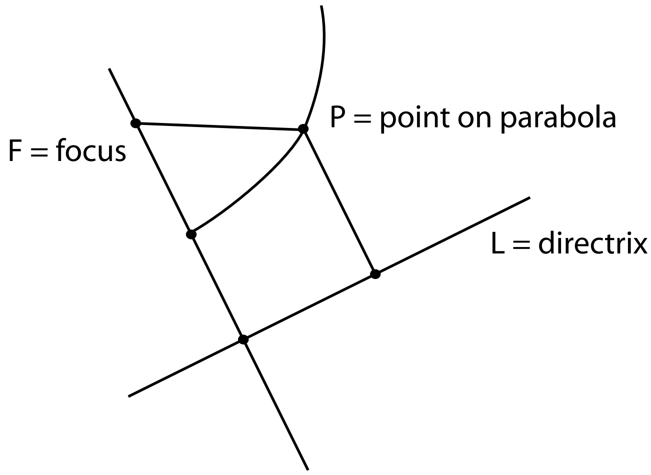

Figure 1: Classical Description of a Parabola

then be half the distance between Q and F. To derive the relation $y = ax^2 + c$, draw a line L' parallel to the directrix passing through a point P on the parabola. Let Q' be the intersection of this line and the extended line QF (see diagram below). The distance between Q' and F is then $|y - 2c|$, and the distance between Q' and P is x. Applying the Pythagorean theorem on the triangle $FQ'P$ then yields

$$|FQ'|^2 + |Q'P|^2 = |PF|^2 = |PX|^2$$
$$|y - 2c|^2 + x^2 = y^2$$

Expanding and simplifying then yields $y^2 - 4cy + 4c^2 + x^2 = y^2$ from which we have $4cy = x^2 + 4c^2$ or $y = \frac{1}{4c} x^2 + c$.

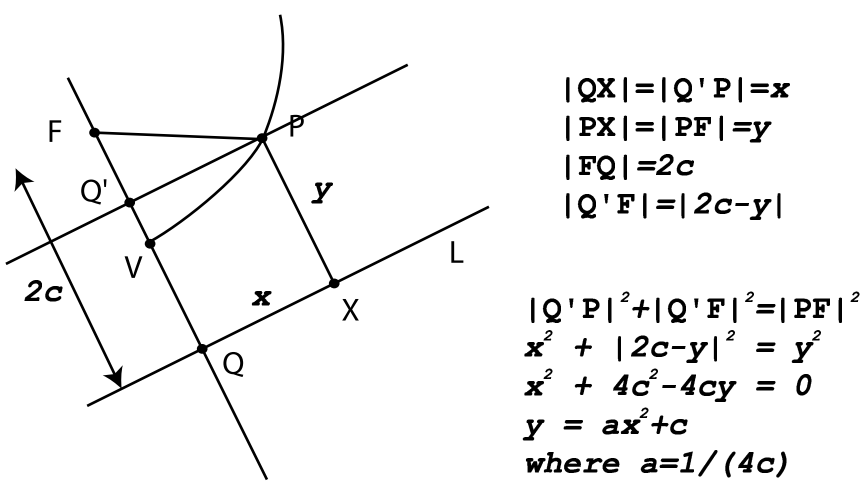

Figure 2: Converting Classical Description to Analytic Equation

The general form of a parabola as a vector valued function can be obtained from the $y = ax^2 + c$ by using vectors to perform the geometric construction described above. The directrix may be described as $X = X_0 + A\,t$. A line perpendicular to the directrix through a point X_1 is described as $X_1 + J(A)\,s$ where $J(A)$ is a the rotation of A by a right angle. (If $A = [a_1, a_2]$ then $J(A) = [-a_2, a_1]$.) We can find the point Q by solving $X_0 + A\,t = F + J(A)\,s$, and then using Q to describe the point $P = Q + A\,x + J(A)\,y$, where A is a unit vector. We have $P = Q + A\,x + J(A)\,(ax^2 + c)$ or $P = (Q + J(A)\,c) + A\,x + aJ(A)\,x^2$ which is of the form $c(t) = \mathbf{a}_0 + \mathbf{a}_1\,t + \mathbf{a}_2\,t^2$. It is worth noting that any parabola can be put in vector-form.

7.2 Constructing a Parabola

The geometric definition gives one method for constructing a parabola, but in geometric modelling we are more likely to want to construct a parabolic arc starting at one point and ending at another. We need additional information to construct a parabolic arc passing through the two endpoints of the parabolic arc. But how much more information is needed and what information is useful to prescribe?

The geometric construction provides a clue as to how much more information is needed. The focus-directrix definition of a parabola requires three non-collinear points; a focus, and two points to define the directrix. However, one can use less information to describe the directrix as a line can technically be defined by one point and a direction (a unit vector). Therefore, to construct a parabolic arc, we need to specify at least two points (one to specify the focus and one the directrix), a unit vector (to specify the directrix completely) and two numbers to specify the starting point and ending point on the segment. This means at a minimum counting the amount of information in terms of coordinates and numbers, we need to specify seven values a little more than three points in a plane. In particular, we can not specify only three points and get a unique parabola.

It is very important to note that there is not one unique parabola through any three points. A basic theorem in projective geometry states that five points are required to form a conic section, either a parabola, an ellipse or a hyperbola, but which type of conic section is not known until all five points are known. We note here that there is actually a one parameter family of parabolas that pass through any three points, as we shall see when we discuss the parametric form of the interpolation problem.

An elementary construction of a parabola can be achieved from three points by exploiting some facts about the geometry of a parabola. This construction fixes one of the parabolas by using some specific properties of parabolas. Specifically, we use the fact there are no parallel tangent lines to a parabola; given a line l there is a unique point on a parabola with a tangent line parallel to l, at least as long as l is not perpendicular to the directrix. A parabola can be then uniquely determined by specifying two points on the parabola and the point of intersection of the tangent lines at these points. Notice that these three points must be non-collinear and form a triangle if they are to create a parabola.

To construct a parabola in this manner, we let $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{p}_2$ be the vertices of the triangle, with $\mathbf{p}_0$ and $\mathbf{p}_2$ the given points on the parabola and $\mathbf{p}_1$ the intersection of the tangents at $\mathbf{p}_0$ and $\mathbf{p}_2$. The tangent lines at $\mathbf{p}_0$ and $\mathbf{p}_2$ are described by the lines from p_0 to p_1 and p_1 to p_2 by

$$l_0(t) = (1-t)\,\mathbf{p}_0 + t\,\mathbf{p}_1 \quad \text{and} \quad l_1(t) = (1-t)\,\mathbf{p}_1 + t\,\mathbf{p}_2.$$

A parabola can be obtained by defining

$$\mathbf{c}(t) = (1-t)\,l_0(t) + t\,l_1(t) = (1-t)^2\,\mathbf{p}_0 + 2t(1-t)\,\mathbf{p}_1 + t^2\,\mathbf{p}_2$$

Notice that, when $t = 0$ we have $\mathbf{c}(0) = \mathbf{p}_0$ and when $t = 1$ we have $\mathbf{c}(1) = \mathbf{p}_2$. Moreover, differentiating to construct the tangent lines, we have

$$\mathbf{c}'(t) = 2(1-t)\,(\mathbf{p}_1 - \mathbf{p}_0) + 2t\,(\mathbf{p}_2 - \mathbf{p}_1)$$

so the tangent vectors at $\mathbf{p}_0$ and $\mathbf{p}_2$ are $2(\mathbf{p}_1 - \mathbf{p}_0)$ and $2(\mathbf{p}_2 - \mathbf{p}_1)$ respectively as desired. In particular, we find that $\mathbf{p}_1$ is the intersection of the two tangent lines.

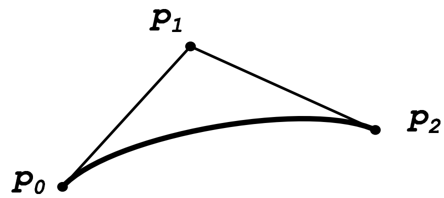

Figure 3: A Geometric Construction from Three Points

This construction is a precursor of a more general construction that we will be using in the remainder of the course, called de Casteljau's algorithm. The general construction entailed in de Casteljau's algorithm will be our first method that using points that are not directly specifying geometric information, but controls the construction. At this moment, the point $\mathbf{p}_1$ is not interpolated directly but does encode geometric information that is interpolated (the prescribed tangent lines) at $\mathbf{p}_0$ and $\mathbf{p}_2$.

It should be noted that this construction though seemingly violating the information requirements set out in the first paragraphs of this subsection does not violate the information requirements. There are three points plus two parameters encoded into the parameterization. This is in addition to the fact that one point specifies the tangent line at the other two points, which adds additional information encoded in the construction.

7.3 Interpolating Parabolas

Let us now consider the problem of determining a parabola that passes through a fixed number of points. The first question that must be considered is the number of points needed. We need three points to determine a parabola. This is a direct result of the algebraic description of a parabola with three unknown vectors. Each vector is determined by one point.

The vector valued description of parabola is a parametric description, so the correct problem is to solve a parametric interpolation problem. Give three points and three parameters, we thus seek the parameteric description of a parabola $c(t)$ with $c(t_0) = p_0$, $c(t_1) = p_1$, and $c(t_2) = p_2$. Writing the equations, we have

$$\begin{aligned}
\mathbf{p}_0 &= \mathbf{a}_0 + \mathbf{a}_1\,t_0 + \mathbf{a}_2\,t_0^2 \\
\mathbf{p}_1 &= \mathbf{a}_0 + \mathbf{a}_1\,t_1 + \mathbf{a}_2\,t_1^2 \\
\mathbf{p}_2 &= \mathbf{a}_0 + \mathbf{a}_1\,t_2 + \mathbf{a}_2\,t_2^2
\end{aligned}$$

Solving these equations for $\mathbf{a}_0$, $\mathbf{a}_1$ and $\mathbf{a}_2$, we apply Gaussian elimination to remove $\mathbf{a}_0$ by looking at the expressions for $\mathbf{p}_0 - \mathbf{p}_1$ and $\mathbf{p}_0 - \mathbf{p}_2$;

$$\begin{aligned}
\mathbf{p}_0 - \mathbf{p}_1 &= \mathbf{a}_1\,(t_0 - t_1) + \mathbf{a}_2\,(t_0^2 - t_1^2) \\
\mathbf{p}_0 - \mathbf{p}_2 &= \mathbf{a}_1\,(t_0 - t_2) + \mathbf{a}_2\,(t_0^2 - t_2^2)
\end{aligned}$$

Factoring these equations, we have

$$\begin{aligned}\mathbf{p}_0 - \mathbf{p}_1 &= (t_0 - t_1)\left(\mathbf{a}_1 + \mathbf{a}_2\,(t_0 + t_1)\right)\\ \mathbf{p}_0 - \mathbf{p}_2 &= (t_0 - t_2)\left(\mathbf{a}_1 + \mathbf{a}_2\,(t_0 + t_2)\right)\end{aligned}$$

We can rewrite these equations and eliminate $\mathbf{a}_1$ to get

$$\frac{\mathbf{p}_0 - \mathbf{p}_1}{t_0 - t_1} - \frac{\mathbf{p}_0 - \mathbf{p}_2}{t_0 - t_2} = \mathbf{a}_2\,(t_1 - t_2)$$

Therefore, simplifying the left-hand side of the above equation, we have

$$\mathbf{a}_2 = \frac{1}{(t_0 - t_1)(t_0 - t_2)}\,\mathbf{p}_0 + \frac{1}{(t_1 - t_0)(t_1 - t_2)}\,\mathbf{p}_1 + \frac{1}{(t_2 - t_0)(t_2 - t_0)}\,\mathbf{p}_2.$$

Back-substituting yields

$$\mathbf{a}_1 = -\frac{t_1 + t_2}{(t_0 - t_1)(t_0 - t_2)}\,\mathbf{p}_0 - \frac{t_0 + t_2}{(t_1 - t_0)(t_1 - t_2)}\,\mathbf{p}_1 - \frac{t_1 + t_0}{(t_2 - t_0)(t_2 - t_1)}\,\mathbf{p}_2$$

and

$$\mathbf{a}_0 = \frac{t_1 t_2}{(t_0 - t_1)(t_0 - t_2)}\,\mathbf{p}_0 + \frac{t_0 t_2}{(t_1 - t_0)(t_1 - t_2)}\,\mathbf{p}_1 + \frac{t_0 t_1}{(t_2 - t_0)(t_2 - t_0)}\,\mathbf{p}_2.$$

Therefore the interpolating polynomial can be written as

$$c(t) = L_0^2(t)\,\mathbf{p}_0 + L_1^2(t)\,\mathbf{p}_1 + L_2^2(t)\,\mathbf{p}_2$$

with

$$\begin{aligned}L_0^2(t) &= \frac{t_1 t_2 - (t_1 + t_2)\,t + t^2}{(t_0 - t_1)(t_0 - t_2)} = \frac{(t - t_1)(t - t_2)}{(t_0 - t_1)(t_0 - t_2)}\\ L_1^2(t) &= \frac{t_0 t_2 - (t_0 + t_2)\,t + t^2}{(t_1 - t_0)(t_1 - t_2)} = \frac{(t - t_0)(t - t_2)}{(t_1 - t_0)(t_1 - t_2)}\\ L_2^2(t) &= \frac{t_0 t_1 - (t_0 + t_1)\,t + t^2}{(t_2 - t_0)(t_2 - t_0)} = \frac{(t - t_0)(t - t_1)}{(t_2 - t_0)(t_2 - t_1)}\end{aligned}$$

Notice that the coefficients of $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{p}_2$ are quadratic polynomials. Moreover the coefficient function $L_i^2(t)$ of p_i has the property that

$$L_i^2(t_j) = \begin{cases}1 & \text{if } i = j,\\ 0 & \text{otherwise.}\end{cases}$$

The 2 in the functions above stand for 2nd order polynomial. These *basis* functions generalize to higher order polynomials (to be discussed later). Assuming that $t_0 < t_1 < t_2$, the basis functions $L_i^2(t)$ graphs appear somewhat like in the diagram below.

One can use matrix algebra to solve the same system of equations by first writing the original system in matrix form as

$$\begin{bmatrix}\mathbf{p}_0 & \mathbf{p}_1 & \mathbf{p}_2\end{bmatrix} = \begin{bmatrix}\mathbf{a}_0 & \mathbf{a}_1 & \mathbf{a}_2\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ t_0 & t_1 & t_2\\ t_0^2 & t_1^2 & t_2^2\end{bmatrix}.$$

The solution is then obtained as

$$\begin{bmatrix}a_0 & a_1 & a_2\end{bmatrix} = \begin{bmatrix}p_0 & p_1 & p_2\end{bmatrix}\begin{bmatrix}\mathbf{p}_0 & \mathbf{p}_1 & \mathbf{p}_2\end{bmatrix} = \begin{bmatrix}\mathbf{a}_0 & \mathbf{a}_1 & \mathbf{a}_2\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ t_0 & t_1 & t_2\\ t_0^2 & t_1^2 & t_2^2\end{bmatrix}^{-1}$$

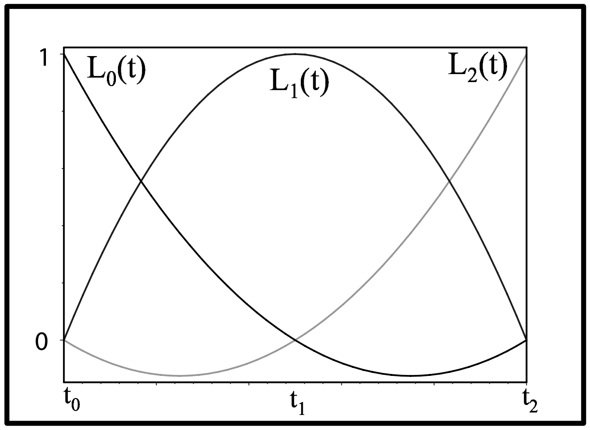

Figure 4: Basis Functions $L_i^2(t)$ for Interpolating Parabolas

Provided that all the parameter values are different the matrix is invertible. The parabola can thus be represented as

$$c(t) = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 & \mathbf{p}_2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ t_0 & t_1 & t_2 \\ t_0^2 & t_1^2 & t_2^2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ t \\ t^2 \end{bmatrix}.$$

In this formulation, one only has to invert the matrix. The solution arrived at yields the same answer (from a theoretical point of view) as the basis function formulation. The advantage of the matrix point of view is that in generalizes easily to higher degree equations, as we really solved the matrix equation $P = AT$ where P is the matrix of data points, A is the matrix of coefficients, and T is the matrix of time values raised to the appropriate powers; the column vectors of T are $[1, t, t^2, \cdots, t^n]$ for an nth order polynomial.

We note that unlike the cases of straight lines and circular curves considered in the previous chapter different parameter values yield different parabolas. This is an important revelation, as the curve is formed from the parameterization not the points.

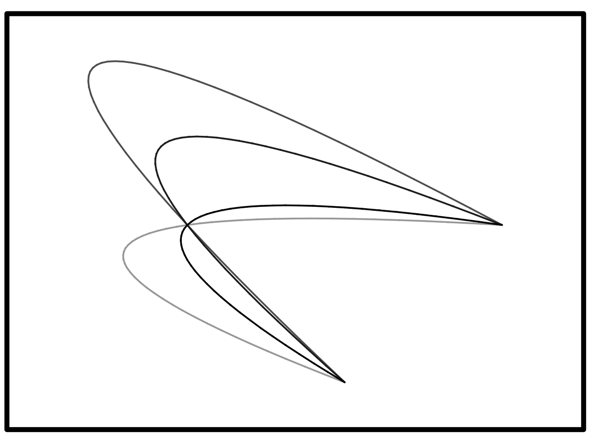

Figure 5: Different Parabolas Through the Same Points

7.4 Piecewise Parabolic Curves

The elementary construction from two points on the parabola and the point of intersection of the tangent lines at these two points allows one to construct a curve consisting or a collection of parabolic arcs, by specifying a list of an odd number of points $\{p_i\}_{i=0}^{2n}$ and forming the triangles $p_{2i-2}p_{2i-1}p_{2i}$ where $i = 1, 2, \cdots, n$. The number n is the number of parabolic segments used in the curve. In general, such a construction will not yield a smooth curve as a discernable angle will be generated at the joint points p_{2i} unless the points p_{2i-1}, p_{2i}, p_{2i+1} are collinear with p_{2i} between p_{2i-1} and p_{2i+1}. The condition that p_{2i-1}, p_{2i}, p_{2i+1} are collinear with p_{2i} between p_{2i-1} and p_{2i+1} ensures that there is a well-defined tangent line at each point on the curve, and thus the curve is geometrically smooth (of order 1).

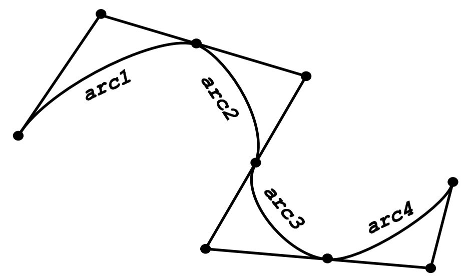

Figure 6: Smooth Curve Constructed out of Parabolic Arcs

Note the derivative is not necessarily defined at the joint point p_{2k} with $k = 1, 2, \cdots, (n-1)$ even when p_{2i-1}, p_{2i}, p_{2i+1} are collinear with p_{2i} between p_{2i-1} and p_{2i+1}. Therefore a piecewise parabolic curve is not necessarily parametrically smooth. To show this, consider a piecewise parabolic curve $c(t)$ of two segments with $c(0) = p_0$, $c(1) = p_2$ and $c(2) = p_4$. A unique tangent line is implied by having p_1, p_2, p_3 collinear, but this condition for geometric smoothness does not imply parametric smoothness for by the calculations above the derivative when $t = 1$ are given by

$$\lim_{t->1^+} c'(t) = 2(p_2 - p_1) \quad \text{and} \quad \lim_{t->1^-} c'(t) = 2(p_3 - p_2).$$

These quantities are equal only if $p_2 = \frac{1}{2}(p_3 + p_1)$, that is if p_1 is the midpoint of the segment p_1p_3. However, because p_1, p_2, p_3 are collinear with p_2 between p_1 and p_3, the vectors $p_2 - p_1$ and $p_3 - p_2$ are parallel and point in the same direction which implies that the tangent line at p_1 is defined even the derivative is not. This is an aspect that is important in geometric modelling and in the construction algorithms we will consider later in the course.

It is worth noting that there is a parameterization of the parabola that makes the piecewise curve parametrically smooth. One needs to consider the generic problem $c(t_0) = p_0$, $c(t_1) = p_2$, $\cdots$, $c(t_n) = p_{2n}$, and then choose values of t_i that make the curve differentiable when $t = t_i$ with $i = 1, ..., n-1$. This is left as an exercise.

7.5 Geometric Smoothness versus Parametric Smoothness

We have just showed the difference between parametric smoothness and geometric smoothness for parabola segments using the first derivative. This is smoothness of order one because we have only used the first derivative. There are higher degrees of smoothness associated with the higher order derivatives. To distinguish between these different types of smoothness, we use C^k to represent parametric smoothness of order k and G^k to represent geometric smoothness of order k.

We note that in general geometric smoothness allows extra degrees of freedom over parametric smoothness. For piecewise parabolic curve consisting of n parabolic arcs, let $\{p_i\}$ $i = 0, 1, 2, \cdots, 2n$ be the control points. A curve that is C^1 (parametrically smooth) has the control points p_{2i} completely free plus the control point p_1. The remaining control points are determined by the C^1 condition $p_{2i+1} - p_{2i} = p_{2i} - p_{2i-1}$. This means that p_1 is completely free, but the remaining control points are fixed. A curve that is G^1 (geometrically smooth) has the control points p_{2i} completely free, and the remaining control points are only constrained to by the condition that p_{2i-1}, p_{2i}, p_{2i+1} are collinear with p_{2i} between p_{2i-1} and p_{2i+1}. This means that p_1 is again completely free, but the remaining control points have one degree of freedom as they are constrained to a line but the position is not determined exactly.

7.6 Fitting Parabolas to Data

So far, we have discussed various methods for creating parabolas, mainly through an interpolation process. In this section, we want to consider a different problem; finding a parabola that fits a set of data points. We accomplish this by applying the method of least squares to finding the parabola that is closest to a set of data.

We concentrate on the parametric form of this problem. Given data (t_i, q_i) with $i = 0, 1, 2, \cdots, n$ find the parabola that is the best fits the data. This means find the "best" solution to set of linear equations

$$\begin{aligned} q_0 &= a_0 + a_1\, t_0 + a_2\, t_0^2 \\ q_1 &= a_0 + a_1\, t_1 + a + 2\, t_1^2 \\ \vdots &= \vdots + \vdots + \vdots \\ q_n &= a_0 + a_1\, t_n + a_2\, t + n^2 \end{aligned}$$

We apply the method of least squares to solve this problem. The problem involves the matrix version of the method of least squares. Let Q be the matrix with row vectors q_i, A be the matrix with row vectors a_j and T be the matrix with row vectors $[1, t_i, t_i^2]^t$. We then apply the method of least squares to solve

$$Q = T\, A$$

for A. Therefore, we multiply by T^t on the right obtaining

$$T^t\, Q = (T^t\, T)\, A$$

The matrix $T^t\, T$ is a square matrix (3×3) and invertible, therefore $A = (T^t\, T)^{-1}\, T^t\, Q$. The fact that it is invertible is a result of the vectors $[1, t_i, t_i^2]$ linearly independent as long as t_i are distinct.

EXAMPLE: Consider the data points with parameter values t and s in the table below.

t	-1.00	0.50	1.20	1.80	2.50
x	-0.79	1.18	1.29	0.93	0.16
y	2.38	0.94	0.61	0.55	0.68

To find the parabola in the form $c(t) = \mathbf{a}_0 + \mathbf{a}_1\, t + \mathbf{a}_2\, t^2$, we need to apply least squares to solve the system

$$\begin{bmatrix} -0.79 & 2.38 \\ 1.18 & 0.94 \\ 1.29 & 0.61 \\ 0.93 & 0.55 \\ 0.16 & 0.68 \end{bmatrix} = \begin{bmatrix} 1.00 & -1.00 & 1.00 \\ 1.00 & 0.50 & 0.25 \\ 1.00 & 1.20 & 1.44 \\ 1.00 & 1.80 & 3.24 \\ 1.00 & 2.50 & 6.25 \end{bmatrix} \begin{bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix}$$

Solving for $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{a}_2$ by the method of least squares we find,

$$\begin{bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix} = \begin{bmatrix} 0.77 & 1.29 \\ 1.04 & -0.85 \\ -0.52 & 0.24 \end{bmatrix}.$$

Thus, the parabola is $x = 0.77 + 1.04\, t - 0.52\, t^2$ and $y = 1.29 - 0.85\, t^2 + 0.24\, t^2$.

In a data fitting problem, generally one is not given the parameter values. The parameter values must be chosen. There are several responsible choices. The first is to choose equally spaced parameter values, choose a increment value Δt, and define $t_i = t_0 + i\Delta t$ for $i = 1, 2, \cdots, n$. A second reasonable choice is to use the chord length values. The chord lengths are the distance $d_i = \|p_{i+1} - p_i\|$. Set $t_0 = 0$ and inductively define $t_{i+1} = t_i + d_i$. A justification for using chord length is that it takes into account some of the natural geometry of the data points. A third possibility is to use centripetal spacing; set $t_0 = 0$ and inductively define $t_{i+1} = t_i + d_i^{1/2}$. Centripetal spacing will smooth out variations in the centripetal (or normal) acceleration of the curve.

EXAMPLE: Consider the problem of finding a parabola to the data below (points only)

x	-0.26	0.80	2.40	3.34	3.92	5.76	7.01	8.33
y	0.47	-0.28	-0.22	-0.23	-0.08	0.52	0.82	0.72

In the diagrams below, we show the best fit parabolas obtained by least squares using the equally spaced parameters and chord length parameters.

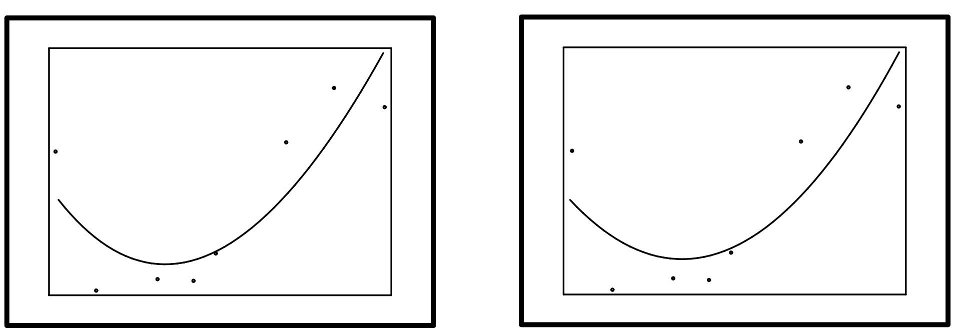

Figure 7: Fitted Parabolas, Left Figure uses Equally Spaced Parameter Values and Right Figure uses Chord Length Parameter Values

It should be noted that one can always normalize the parameter values so that $t_0 = 0$ and $t_n = 1$, by defining the normalized parameter $s = (t - t_0)/(t_n - t_0)$. The normalized parameter will yield the same curve (up to round-off error). The advantage of the normalized parameter is that it specifies the range of the parameter to the fixed interval $0 \leq s \leq 1$. An additional advantage is that some construction algorithms (Bezier curves in particular) work with normalized parameters.

7.7 Exercises

1. (Computational) Find a parametric equation of a parabola that
   (a) passes through the points $[1, 1]$, $[2, 3]$ and $[3, 2]$ at $t = 0$, $t = 1$, and $t = 3$
   (b) passes through $[1, 2]$ and $[3, 4]$ at $t = 0$ and $t = 2$ and has tangent vector $[4, 3]$ at $t = 0$.
   (c) passes through $[1, 2]$ and $[3, 4]$ at $t = 0$ and $t = 2$ and has tangent vector $[4, 3]$ at $t = 1$.
   (d) passes through $[3, 1]$ at $t = 0$ and has tangent vectors $[2, 2]$ and $[3, 5]$ at $t = 0$ and $t = 1$.

2. (Computational) Given the control points

i	x	y
0	1.0	2.0
1	2.5	1.0
2	3.0	2.0
3	4.0	4.0
4	3.0	5.0

   (a) Construct the piecewise parabolic curve from these control points using the geometric construction with $c(0) = [1, 2]$, $c(1) = [3, 2]$ and $c(2) = [3, 5]$
   (b) Show that this curve is geometrically smooth but not parameterically smooth.
   (c) Find a change of parameterization that makes the curve parametrically smooth, that is $c(t_0) = [1, 2]$, $c(t_1) = [3, 2]$ and $c(t_2) = [3, 5]$.
   (d) Move the control point [3,2] so that the curve constructed from the geometric algorithm is parametrically smooth.

3. (Interactive) Complete the interactive exercises associated with piecewise parabolic curves.

4. Given the data in the table

x	y
-2.0	4.0
-1.0	2.0
0.0	1.0
0.5	1.5
1.0	2.5
2.0	3.0

(a) Find a parabola $y = ax^2 + bx + c$ that best fits the data

(b) Find a parametric parabola that best fits the data using the equal spaced parameter values with increment $\Delta t = 0.1$.

(c) Find a parametric parabola using chord length parameter values

(d) Find a parametric parabola using centripetal spaced parameter values.

(e) Plot each of parabolas generated above, which is provides the best fit to the data in your opinion. Why?