MA 323 Geometric Modelling
Course Notes: Day 31
Blended and Ruled Surfaces
Coons Patches

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Today, we want to start considering patches that are constructed solely by specifying the edge curves. We have seen one such method earlier, Hemite patches with zero twist vector, where we only chose the points on the edge of the patch and then determined the interior points so that there were zero twist vectors on the corners. This method can be extended to arbitrary number of rectangles that were joined together. However, this method has the same problem as a regular rectangular grid.

We want to look at much more general system of blended surfaces, and then Coons patches. These types of patches are given by specifying a system of curves and then constructing a patch by a sweep of curves. In a sense this is a simpler method than Bezier patches. It has the advantage of smooth joining by losing some control on the interior of the patch and possibly the smoothness of the patch on the interior.

31.1 Blended and Ruled Surfaces

The construction of blended surfaces involves solving a simpler problem than constructing a surface patch. We start with the problem of forming a surface by blending two curves. In particular, the problem now is given two curves α_1 and α_2 to construct a surface that passes through the two curves. Suppose we are given two curves $\alpha_1(s)$ and $\alpha_2(s)$ that are parameterized over the same interval. The parameterization of the curve is important. In fact, we will assume the curves are defined over the interval $0 \leq s \leq 1$. Given these curves, we can easily define a surface that connects $\alpha_1(s)$ to $\alpha_2(s)$ by linear interpolates between the curves as

$$x(s,t) = (1-t)\,\alpha_1(s) + t\,\alpha_2(s).$$

Notice all we have done is connected points with the same parameter s with a straight line. This is the simplest type of blended surface, where the blending function is linear. This type of blended surface is called in mathematics a ruled surface, as the lines joining the two curves are called rulings. It is also called a lofted surface in engineering books and older geometric modeling texts.

We can use other types of blending functions. For instance, another type of blended surface is to blend two curves with cubic curves. To accomplish this we give at each endpoint a tangent vector. Then we can use a cubic Hermite curves to blend the two curves, see diagram below.

Given two curves $\alpha_1(s)$ and $\alpha_2(s)$, and vector fields $V_1(s)$ and $V_2(s)$ on the curves $\alpha_1(s)$ and

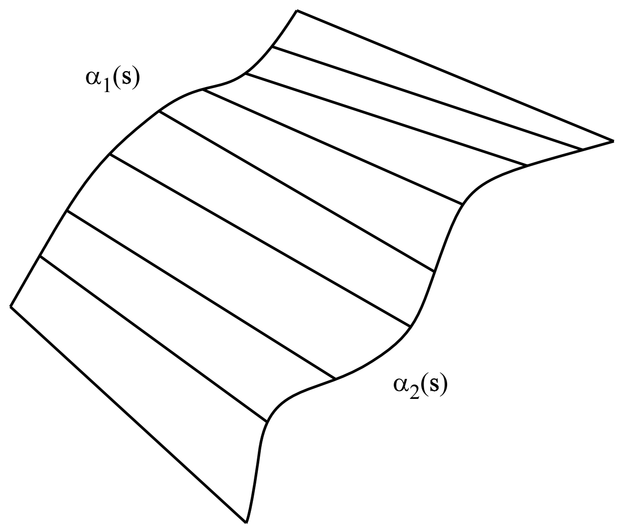

Figure 1: A ruled surface

$\alpha_2(s)$ respectively. Construct the surface

$$x(s,t) = (1-t)^3\,\alpha_1(s) + 3(1-t)^2 t[\alpha_1(s) + \frac{1}{3}\,V_1(s)] + 3(1-t)t^2[\alpha_2(s) + \frac{1}{3}\,V_2(s)] + t^3\alpha_2(s).$$

by using Hermite interpolation in the t direction for each s with $0 \le s \le 1$. Notice that we are using $+$ instead of $-$ for the third term for symmetry. You just interpret $V_2(s)$ as the derivative in the opposite direction. The difference being we have defined the surface so the surface definition is symmetric. It does not matter which curve is α_1 and which is α_2.
It is standard to not give the vector fields $V_1(s)$ and $V_2(s)$ for every s, but rather to give a vector at each of the points $\alpha_1(0)$, $\alpha_1(1)$ and $\alpha_2(0)$, $\alpha_2(1)$. Let these vectors be given by v_{10}, v_{11}, and v_{20}, v_{21} respectively, see diagram below. Then define the vectors $V_1(s) = (1-s)v_{10} + sv_{20}$ and $V_2(s) = (1-s)v_{20} + s\,v_{21}$. This method cuts down on the amount of information that needs to be stored. Other options are to given by construction method of the curve. For instance, if the curves α_i to be blended between are Bezier curves then one can construct the tangent vector fields $V_i(s)$ as Bezier curves by supplying a vector at each control point of the curve.

One can also consider other methods of blending the two curves depending on how the curves $\alpha_1(s)$ and $\alpha_2(s)$ were constructed and depending on the order of complexity desired. For instance, one can blend the curves quintically

31.2 Exercises

1. Given the curves $\alpha_1(s) = [s, 0, \cos(s)]$ and $\alpha_2(s) = [s, 1, \sin(s)]$
   (a) blend the two curves using linear blending.
   (b) blend the two curves using cubic Hermite blending with the constant vectors $T_1(s) = [0, 1, 1]$ and $T_2(s) = [0, -1, 1]$
   (c) blend the two curves using cubic Hermite blending with the variable vectors $T_1(s) = (1-s)[0, 0, 1] + s[0, 1, 1]$ and $T_2(s) = (1-s)[0, -1, 1] + s[1/2, -1/2, 2]$

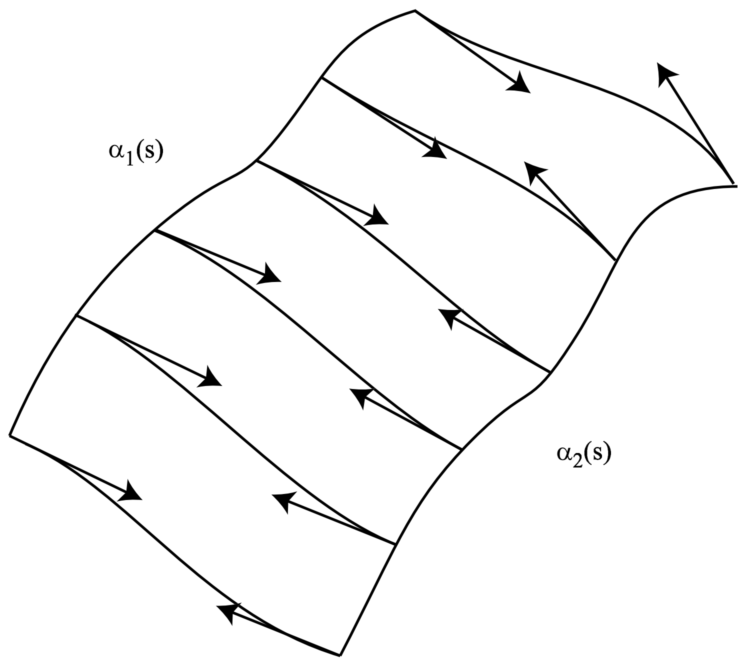

Figure 2: A blended surface

2. To construct a composite blended surface through a sequence of curves α_0, α_1, $\cdots$, α_n. One needs to construct a blended surface between α_{i-1}, α_i for $i = 1, 2, \cdots, n$.
   (a) Is a linearly blended composite surface ever C^1 in the parameters s and t? Under what conditions?
   (b) Is a cubicly blended composite surface ever C^1 in the parameters s and t? Under what conditions is it C^1?
   (c) Is a cubicly blended composite surface ever C^2 in the parameters s and t? Under what conditions is it C^2?

3. How could you construct a C^2 blended composite surface?

4. Write an algorithm to construct a quintically blended composite surface.

31.3 Coons Patches

A Coons patch is a solution to the following interpolation problem:

> Given four curves that intersect at four points. [Let A, B, C, D be four points. Let α_1 be a curve connecting A to B and α_2 be a curve connecting C to D. Let β_1 be a curve connecting A to C and β_2 be a curve connecting B to D.]
> Find a surface that passes through the four curves

The type of surface that arises from solving this type or problem, can be used to construct complex surfaces, as you only have to define curves and blend the curves together. These types of surfaces can be used to form connecting surfaces between two given surfaces, and thus they are used in CAD/CAM systems to blend two cylinders - from an elbow joint.

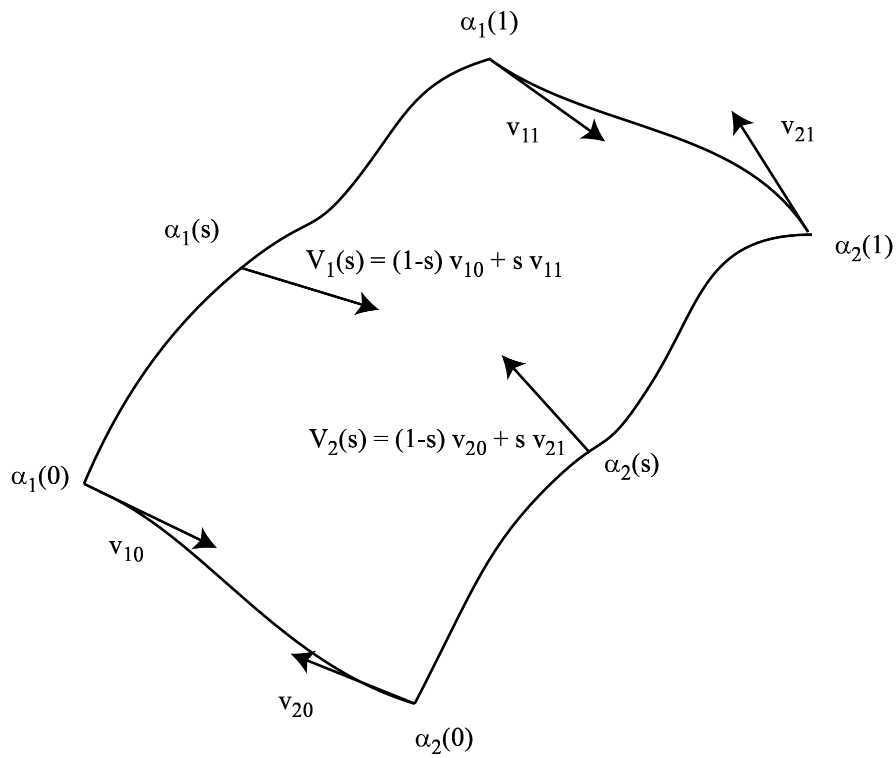

Figure 3: A Standard Method for Constructing Blending

To construct a Coons patch, one blends the curves that correspond to opposite edges, and adjusts the sum of the blendings so that the surfaces pass through the desired edge curves.

The first method that we will use to solve the problem is to use linear blendings between each pair of opposite edges to create two ruled surfaces. Define the surfaces

$$x_1(s,t) = (1-t)\,\alpha_1(s) + t\,\alpha_2(s)$$

and

$$x_2(s,t) = (1-s)\,\beta_1(t) + s\,\beta_2(t).$$

These surfaces satisfy pass through two of the desired edge curves; x_1 has α_1 and α_2 as edge curves and x_2 has β_1 and β_2 as edge curves. To solve the original problem, we combine these two surfaces. It would be nice if we could just add the two surface together to achieve the answer, but that does not work. Geometrically, we want to start at one surface and add a vector to the other surface, that is

$$x(s,t) = x_1(s,t) + \nu_1(s,t) \quad \text{or} \quad x(s,t) = x_2(s,t) + \nu_2(s,t)$$

These deforming vectors $\nu_1(s,t)$ and $\nu_2(s,t)$ need to defined appropriately. We will view these vectors as

$$\nu_1(s,t) = x_2(s,t) - x_3(s,t) \quad \text{or} \quad \nu_2(s,t) = x_1(s,t) - x_3(s,t)$$

where $x_3(s,t)$ is another surface. To understand what surface to use as x_3, we first note that we want $x(s,0) = \alpha_1(s)$. Examining this property for $x(s,t)$, we find from $x(s,t) =$

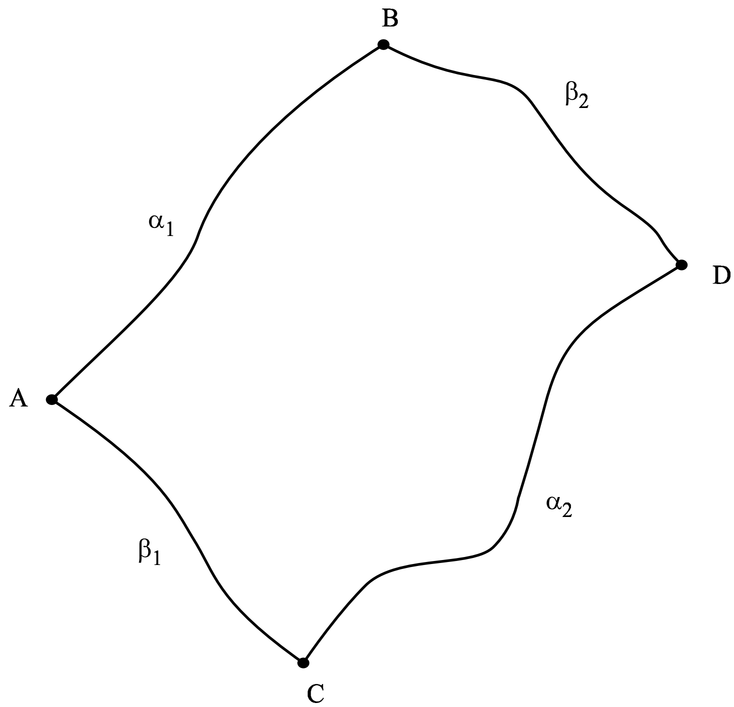

Figure 4: A new interpolation problem

$x_1(s,t) + x_2(s,t) - x_3(s,t)$ that since $x(s,0) = x_1(s,0) = \alpha_1(s)$ that $x_3(s,0) = x_2(s,0)$ so $x_3(s,0)$ needs to linearly interpolate between the corner points $P_{0,0}$ and $P_{1,0}$. Likewise, we find that from $x(s,1) = \alpha_2(s)$ that $x_3(s,1)$ needs to linearly interpolate between the corner points $P_{0,1}$ and $P_{1,1}$. Similar analysis on the edges $x(0,t)$ and $x(1,t)$ reveal that the surface x_3 needs to have $x_3(0,t)$ linearly interpolate between $P_{0,0}$ and $P_{0,1}$ and $x_3(1,t)$ linearly interpolate between $P_{1,0}$ and $P_{1,1}$. Therefore, we need the surface $x_3(s,t)$ to linearly interpolate between all the four corner points, that is bilinear interpolation. Thus, the surface $x_3(s,t)$ is a bilinear patch through the four corner points

$$x_3(s,t) = (1-s)\,[(1-t)\,P_{0,0} + t\,P_{0,1}] + s\,[(1-t)\,P_{1,0} + t\,P_{1,1}].$$

A surface that interpolates between the four edge curves $\alpha_1(s)$, $\alpha_2(s)$, $\beta_1(t)$, $\beta_2(t)$ where $\alpha_1(0) = \beta_1(0) = P_{0,0}$, $\alpha_1(1) = \beta_2(0) = P_{1,0}$, $\alpha_2(0) = \beta_1(1) = P0,1$, $\alpha_2(1) = \beta_2(1) = P_{1,1}$ is

$$\begin{aligned} x(s,t) = {} & (1-t)\,\alpha_1(s) + t\,\alpha_2(s) \\ & + (1-s)\,\beta_1(t) + s\,\beta_2(t) \\ & + (1-s)\,[(1-t)\,P_{0,0} + t\,P_{0,1}] \\ & + s\,[(1-t)\,P_{1,0} + t\,P_{1,1}]. \end{aligned}$$

An example of a bilinear Coons patch is given below.

An alternative method is to use a cubic blending of the opposite pairs of edge curves. For instance, suppose we are given curves $\alpha_1(s)$, $\alpha_2(s)$, $\beta_1(t)$, $\beta_2(t)$ where $\alpha_1(0) = \beta_1(0) = P_{0,0}$, $\alpha_1(1) = \beta_2(0) = P_{1,0}$, $\alpha_2(0) = \beta_1(1) = P_{0,1}$, $\alpha_2(1) = \beta_2(1) = P_{1,1}$. At the corner points, we then define the vectors $T^s_{0,0} = \alpha_1'(0)$, $T^t_{0,0} = \beta_1'(0)$ (T^s is the tangent vector in the s direction and T^t is the tangent vector in the t direction at $P_{0,0}$), $T^s_{1,0} = -\alpha_1'(1)$ and $T^t_{1,0} = \beta_2'(0)$,

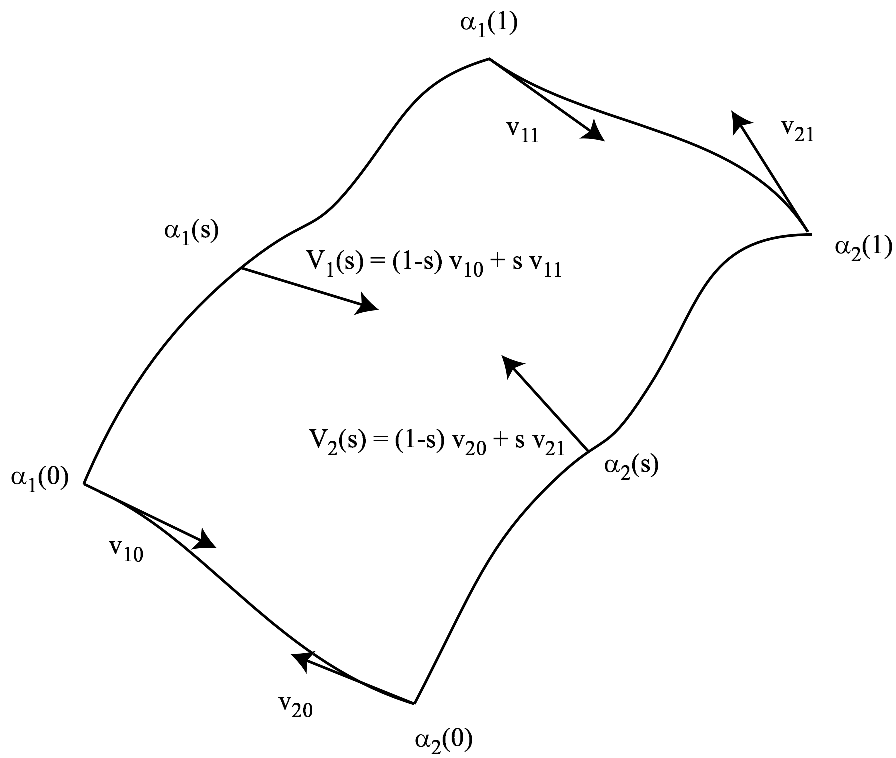

Figure 5: A bilinear blended Coons patch

$T^s_{0,1} = \alpha'_2(0)$ and $T^t_{0,1} = -\beta'_1(1)$, $T^s_{1,1} = -\alpha'_2(1)$ and $T^t_{1,1} = -\beta'_2(1)$ (see diagram below). The minus signs are used whenever 1 is used as a parameter to reflect the symmetry desired in cubicly blending two surfaces.
We then define the surface $x_1(s,t)$ based on blending the curves $\alpha_1(s)$ and $\alpha_2(s)$ in a cubic fashion using the vectors $V_1(s) = (1-s)T^t_{0,0} + s\,T^t_{1,0}$ and $V_2(s) = (1-s)T^t_{0,1} + sT^t_{1,1}$. Likewise, we define the surface $x_2(s,t)$ based on blending the curves $\beta_1(t)$ and $\beta_2(t)$ using the vectors $V_1(t) = (1-t)T^s_{0,0} + tT^s_{0,1}$ and $V_2(t) = (1-t)T^s_{1,0} + tT^s_{1,1}$.

The surface passing through the curves α_1, α_2, β_1 and β_2 is defined using the above surfaces $x_1(s,t)$ and $x_2(s,t)$ by

$$x(s,t) = x_1(s,t) + x_2(s,t) - x_3(s,t)$$

where $x_3(s,t)$ is a surface defined to remove the cubic interpolation curves between $P_{0,0}$, $P_{1,0}$ and between $P_{0,1}$, $P_{1,1}$ introduced in creating $x_1(s,t)$ and the cubic interpolation curves between $P_{0,0}$, $P_{1,0}$ and between $P_{0,1}$, $P_{1,1}$ introduced in creating $x_2(s,t)$. One manner to remove this to perform a bicubic blending on the edge points. Create cubic Hermite curves using the information at the corner points then blend these cubic curves. You will get two different surfaces depending on which direction is blended. You will examine this in detail in one of the exercises.

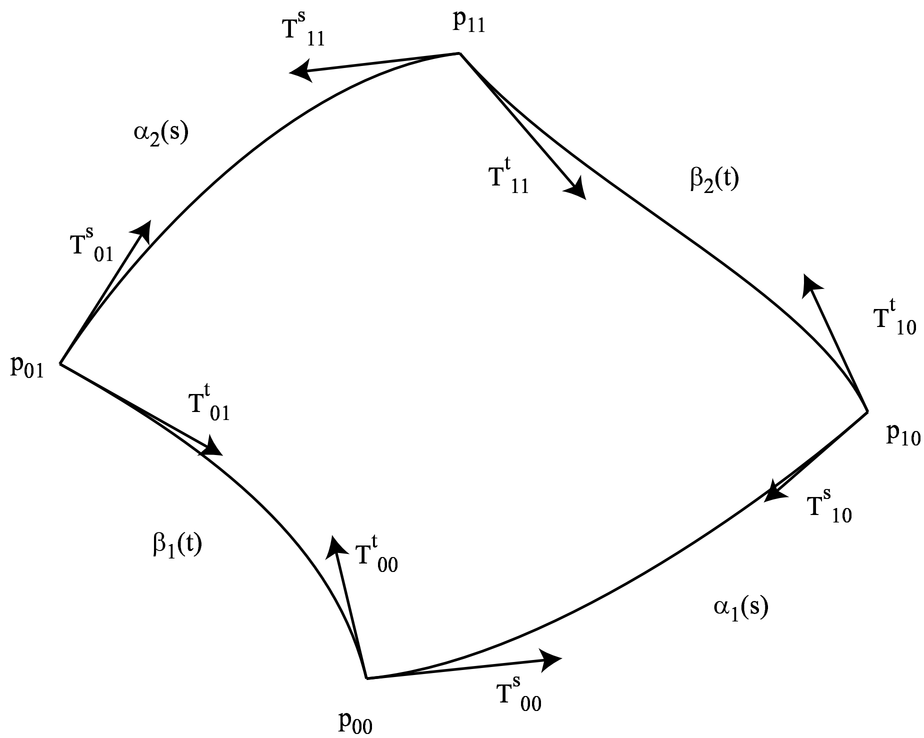

Figure 6: Set-up for a bicubic blended Coons patch

31.4 Exercises

1. Given the curves $\alpha_1(s) = [s, 0, s - s^2]$, $\alpha_2(s) = [s, 1, s]$, $\beta_1(t) = [0, t^2, t - t^2]$, $\beta_2(t) = [1, t, t]$. Notice that $\alpha_1(0) = \beta_1(0)$, $\alpha_1(1) = \beta_2(0)$, $\alpha_2(0) = \beta_1(1)$, $\alpha_2(1) = \beta_2(1)$. Write the Coons patch using linear blending for both curves.

2. Coons patches with Hermite blending functions. Suppose you are given curves $\alpha_1(s)$, $\alpha_2(s)$, $\beta_1(t)$, $\beta_2(t)$ with $A = \alpha_1(0) = \beta_1(0)$, $B = \alpha_1(1) = \beta_2(0)$, $C = \alpha_2(0) = \beta_1(1)$, $D = \alpha_2(1) = \beta_2(1)$. Moreover at the corner points, we are given vector $U_A = \alpha_1'$, V_A, U_B, V_B, U_C, V_C and U_D, V_D.

   (a) Form the surface $x_1(s,t)$ by blending α_1 and α_2 using the vectors $T_1(s) = (1 - s)U_A + sU_B$ and $T_2(s) = (1 - s)U_C + sU_D$.

   (b) Form the surface $x_2(s,t)$ by blending β_1 and β_2 using the vectors $S_1(t) = (1 - t)V_A + tV_C$ and $S_2(t) = (1 - t)V_B + tV_D$.

   (c) Form the surface $x_3(s,t)$ by blending the cubic Hermite curves A, V_A, V_B, B and C, U_C, U_D, D defined in s using the vectors $T_1(s)$ and $T_2(s)$ defined above.

   (d) Define the surface $x(s,t) = x_1(s,t) + x_2(s,t) - x_3(s,t)$.

   (e) Verify that the surface $x(s,t)$ satisfies $x(s,0) = \alpha_1(s)$, $x(s,1) = \alpha_2(s)$, $x(0,t) = \beta_1(t)$, and $x(1,t) = \beta_2(t)$.

   (f) Form the surface $x_4(s,t)$ by blending the cubic Hermite curves A

(g) Define the surface $y(s,t) = x_1(s,t) + x_2(s,t) - x_4(s,t)$ and verify that this surface satisfies $x(s,0) = \alpha_1(s)$, $x(s,1) = \alpha_2(s)$, $x(0,t) = \beta_1(t)$, and $x(1,t) = \beta_2(t)$.

3. Use the method in previous exercise to create the Coons patch with the curves $\alpha_1(s) = [s, 0, s - s^2]$, $\alpha_2(s) = [s, 1, s]$, $\beta_1(t) = [0, t^2, t - t^2]$, $\beta_2(t) = [1, t, t]$. Verify that $x(s,t)$ and $y(s,t)$ generate different curves.

4. How would you define a composite Coons patch?

(a) Describe the smoothness of a linearly blended Coons patch?

(b) Describe the smoothness of a cubicly blended Coons patch?