MA 323 Geometric Modelling
Course Notes: Day 30
Triangular Bezier Patches

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Today, we extend the idea of a Bezier patch to a triangular grid. This allows greater flexibility in some sense of creating surfaces, as it is very easy to create an approximation by triangulization. In fact, there are well-defined methods in computational geometry of constructing triangulizations of set of points or more generally a polygonal grid (a polyhedra). These allow one to create a refinement of any wireframe type mesh of a surface and then overlay a smooth surface.

30.1 Triangular Grids and Barycentric Coordinates

To construct a triangular patch, it is first helpful to review barycentric coordinates for a plane. Given three noncollinear points, A, B, C in a plane. Any other point P is determined by

$$P = \alpha\, A + \beta\, B + \gamma\, C$$

where $\alpha + \beta + \gamma = 1$. Recall if $0 \leq \alpha,\ \beta,\ \gamma \leq 1$ then then the point P lies within the triangle formed by vertices A, B, C.

The construction of a triangular patch is based upon deriving a de Casteljau's type algorithm with recursion based on triangles instead of rectangles. The structure for the control points is defined by a triply indexed set of points $p_{i,j,k}$ with $i + j + k = n$ forming a triangular grid of size n, see the diagram below.

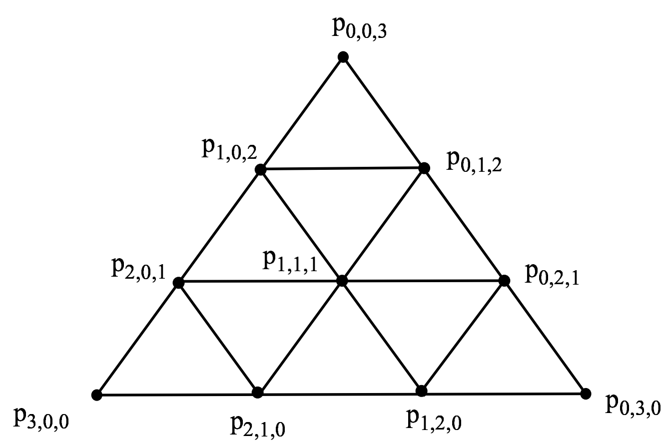

Figure 1: A triangular grid

The recursion is given by defining

$$p_{i,j,k} = u\,p_{i+1,j,k} + v\,p_{i,j+1,k} + w\,p_{i,j,k+1} \quad \text{with } i+j+k = n-m$$

to fill out the triply indexed set of points $p_{i,j,k}$ to the $i+j+k \leq n$. The point $p_{0,0,0}$ is then a point on the patch. The patch is then created by considering all values of the parameters u, v, w with $0 \leq u, v, w \leq 1$ and $u+v+w = 1$.

Example: Consider $n = 2$, and $p_{2,0,0} = [0,0,0]$, $p_{1,1,0} = [1,0,2]$, $p_{0,2,0} = [0,2,0]$, $p_{1,0,1} = [0,1,0]$, $p_{0,1,1} = [1,1,1]$, $p_{0,0,2} = [0,2,1]$. (a) Compute the point on the patch for $u = 1/2$, $v = 1/3$, $w = 1/6$. (b) the patch for arbitrary u, v, w

(a)

$$\begin{aligned}
p_{1,0,0} &= \frac{1}{2}[0,0,0] + \frac{1}{3}[1,0,2] + \frac{1}{6}[0,1,0] = [\frac{1}{3}, \frac{1}{6}, \frac{2}{3}] \\
p_{0,1,0} &= \frac{1}{2}[1,0,2] + \frac{1}{3}[0,2,0] + \frac{1}{6}[1,1,1] = [\frac{2}{3}, \frac{5}{6}, \frac{7}{6}] \\
p_{0,0,1} &= \frac{1}{2}[0,1,0] + \frac{1}{3}[1,1,1] + \frac{1}{6}[0,2,1] = [\frac{1}{3}, \frac{7}{6}, \frac{1}{2}] \\
p_{0,0,0} &= \frac{1}{2}[\frac{1}{3}, \frac{1}{6}, \frac{2}{3}] + \frac{1}{3}[\frac{2}{3}, \frac{5}{6}, \frac{7}{6}] + \frac{1}{6}[\frac{1}{3}, \frac{7}{6}, \frac{1}{2}] = [\frac{4}{9}, \frac{5}{9}, \frac{29}{36}]
\end{aligned}$$

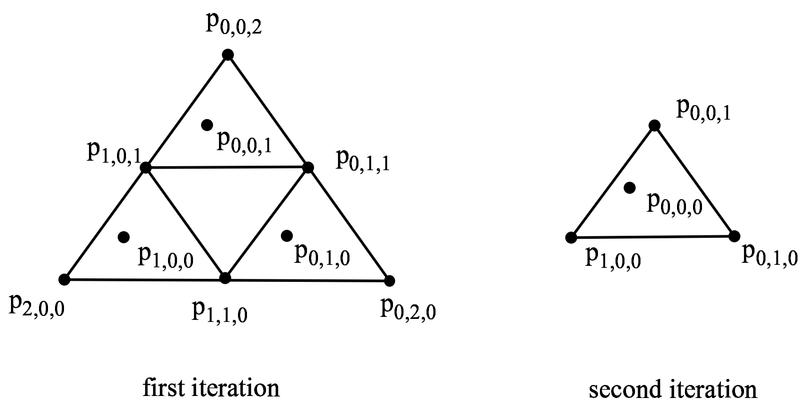

Figure 2: Illustration of de Casteljau's algorithm for triangular grids

(b)

$$\begin{aligned}
p_{1,0,0} &= u[0,0,0] + v[1,0,2] + w[0,1,0] = [v, w, 2v] \\
p_{0,1,0} &= u[1,0,2] + v[0,2,0] + w[1,1,1] = [u+w, 2v+w, 2u+w] \\
p_{0,0,1} &= u[0,1,0] + v[1,1,1] + w[0,2,1] = [v, u+v+2w, v+w] \\
p_{0,0,0} &= u\,[v, w, 2v] + v\,[u+v, 2v+w, 2u+w] + w\,[v, u+v+2w, v+w] \\
&= [2uv + v^2 + vw, 2uw + 2vw + 2v^2 + 2w^2, 4uv + 2vw + w^2]
\end{aligned}$$

Working through the recursion relation for generic points, one can show that the patch can be written using a Bernstein polynomial type form as

$$X(u,v,w) = \sum_{i+j+k=n} B^n_{i,j,k}(u,v,w)\,p_{i,j,k}$$

where

$$B^n_{i,j,k} = \frac{n!}{i!j!k!}\, u^i\, v^j\, w^k.$$

The advantage of having such a Bernstein representation for triangular patches is for joining patches smoothly. One can differentiate across the three edge curves defined respectively by $u = 0$, $v = 0$ and $w = 0$. The conditions on the control points are not easy to derive, and we will not go into great detail into the derivation, which is left as a challenging exercise, rather we will just state the conditions.

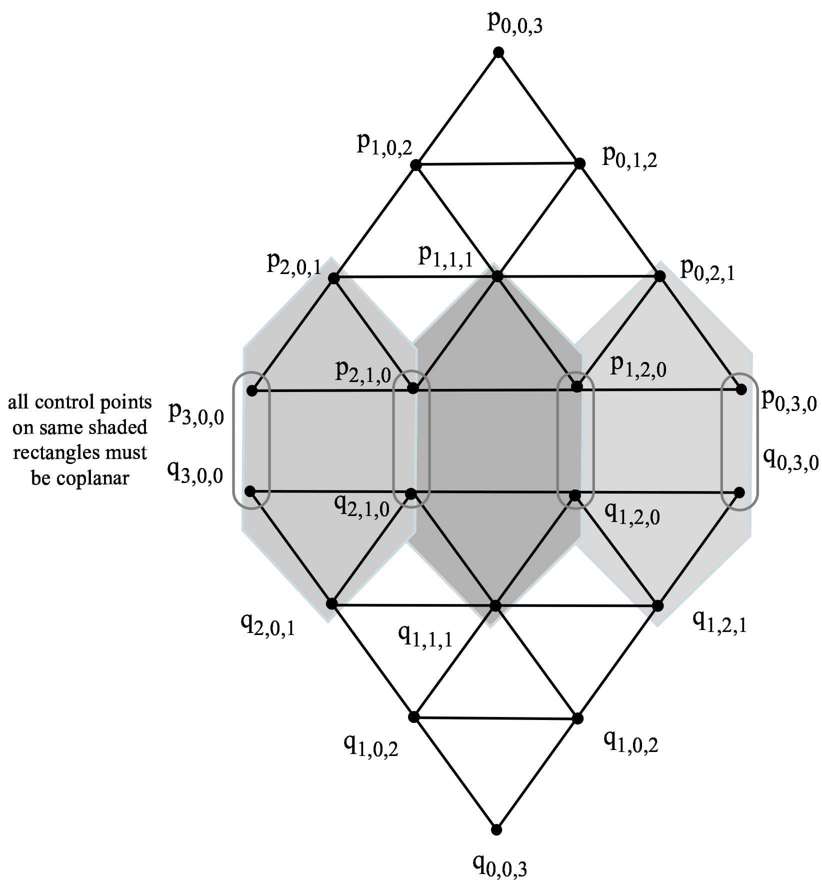

Figure 3: Compatibility conditions for smooth joining across a single edge

First, we note that the tangent plane from the de Casteljau's algorithm for a triangular grid is given by the plane through the points $p_{1,0,0}(u, v, w)$, $p_{0,1,0}(u, v, w)$ and $p_{0,0,1}(u, v, w)$. This implies that for two triangular patches meeting on the edge $w = 0$ (in both representations, see diagram above) that the points $p_{1,0,0}(u, v, 0) = q_{1,0,0}(u, v, 0)$, $p_{0,1,0}(u, v, 0) = q_{0,1,0}(u, v, 0)$, $p_{0,0,1}(u, v, 0)$ and $q_{0,0,1}(u, v, 0)$ should be coplanar. An examination of the recursion formula shows that $p_{0,0,1}(u, v, 0)$ is a point on the Bezier curve of degree $n - 1$ formed by the points $p_{i,j,1}$ with $i + j = n - 1$, $p_{1,0,0}(u, v, 0)$ is a point on the Bezier curve of degree $n - 1$ formed by points $p_{i+1,j,0}$ with $i + j = n - 1$, $p_{0,1,0}(u, v, 0)$ is a point on the Bezier curve of degree $n - 1$ formed by points $p_{i,j+1,0}$ with $i + j = n - 1$. To achieve the coplanarity for smooth joining across an edge, it is sufficient that each set of four points

$p_{i+1,j,0} = q_{i+1,j,0}$, $p_{i,j+1,0} = q_{i,j+1,0}$, $p_{i,j,1}$, $q_{i,j,1}$ be coplanar with $i + j = n - 1$, see diagram above.

The advantage of triangular patches is in their use for defining composite patches. This advantage comes from the fact that one can always define a triangular grid on an object very simply. In general, triangular patches are useful once one has a *topological* handle on the object to be modelled. The topology of a surface s determined by the admission of a triangulation or a grid of triangles on the object. It is always possible to triangulate an object, but it is not always possible to form a rectangular grid on an object. We will discuss topological constructs in the next few weeks, when we consider subdivision surfaces.

30.2 Exercises

1. Compute the point on the patch with $u = 1/3$, $v = 1/4$, $w = 5/12$ for the patch with control points given by

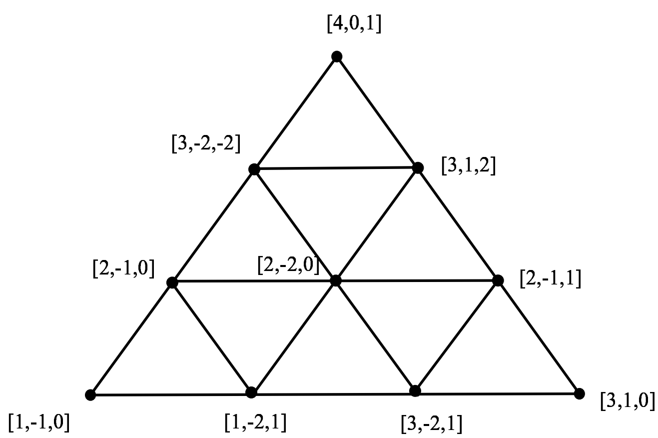

Figure 4: Apply de Casteljau's algorithm to the control points

2. Verify that when $w = 0$, $u = t$, $v = 1 - t$ the point lies on a Bezier curve with control points $q_i = p_{n-i,i,0}$ by working through the de Casteljau's type algorithm.

3. Show that the tangent plane to a triangular Bezier patch $X(u, v, w)$ is given by the plane through the points $p_{1,0,0}(u, v, w)$, $p_{0,1,0}(u, v, w)$ and $p_{0,0,1}(u, v, w)$ obtained from de Casteljau's algorithm.

4. Use the triangular Bezier patch applet to investigate shape of quadratic triangular patches and cubic triangular patches.

5. Use the dual triangular Bezier patch applet to investigate the joining of two triangular Bezier patches, and verify the condition given for smoothness visually.

6. *Challenge Exercise:* For the two triangular Bezier patches of degree n with control points $p_{i,j,k}$ and $q_{i,j,k}$ that meet along the edge $p_{i,j,0} = q_{i,j,0}$, show that to join for the patches to meet geometrically smoothly (G^1) it is sufficient that the control points $p_{i,j,1}$, $p_{i+1,j,0} = q_{i+1,j,0}$, $p_{i,j+1,0} = q_{i,j+1,0}$ and $q_{i,j,1}$ are coplanar.,