MA 323 Geometric Modelling
Course Notes: Day 20
Curvature and G^2 Bezier splines

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Yesterday, we introduced the notion of curvature and how it plays a role formally in the description of curves, as a geometric second derivative. Today, we want to continue this discussion and construct curves that have second order continuity. In this manner, we derive the curvature of a cubic Bezier curve at the endpoints only. It is worth noting that since cubic curves are smooth curves, the curvature (when defined, as it is possible to construct a cusp with a cubic curve) is continuously varying. In the construction of a G^2 spline, it is therefore only necessary to match the curvatures are the endpoints.

20.1 G^2 Bezier splines

A G^2 spline is a piecewise cubic Bezier curve which is G^1 and the derivative is G^1. They are much more difficult to describe analytically, through control points and the functional representation. Basically, we want to reparameterize the curve with respect to the arclength parameter and show the curve is C^2 respect to the arc-length parameter, which means the curvatures and the unit tangent vectors must be equal at the joint point.

Let p_0^1, p_1^1, p_2^1, p_3^1 and p_0^2, p_1^2, p_2^2, p_3^2 be the control point of two cubic Bezier curves. We want a continuous curve so $p_3^1 = p_0^2$. We want the curve G^1, so $p_3^1 = p_0^2$ lies on the segment $p_2^1 p_1^2$. Further, let d be the intersection of the lines $p_1^1 p_2^1$ and $p_1^2 p_2^2$, a C^2 spline control point controlling the location of the joint point. To derive the conditions for G^2, let $r_- = \text{ratio}(p_1^1, p_2^1, d)$, $r_+ = \text{ratio}(d, p_1^2, p_2^2)$, and $r = \text{ratio}(p_2^1, p_3^1, p_1^2)$. We show below that the curvatures at $p_3^1 = p_0^2$ are given by

$$\kappa_- = \frac{4}{3}\frac{\text{area}(p_1^1, p_2^1, p_3^1)}{\|p_3^1 - p_2^1\|^3} \qquad \text{and} \qquad \kappa_+ = \frac{4}{3}\frac{\text{area}(p_0^2, p_1^2, p_2^2)}{\|p_1^2 - p_0^2\|^3},$$

from the control points p_0^1, p_1^1, p_2^1, p_3^1 and p_0^2, p_1^2, p_2^2, p_3^2 respectively. Using the below figure, if the curvatures agree then

$$\frac{\text{area}(p_1^1, p_2^1, p_3^1)}{\text{area}(p_2^1, p_3^1, d)} = r^3,$$
$$\frac{\text{area}(p_1^1, p_2^1, p_3^1)}{\text{area}(d, p_0^2, p_1^2)} = r_-,$$
$$\frac{\text{area}(p_0^2, p_1^2, p_2^2)}{\text{area}(p_2^1, p_3^1, d)} = r_+,$$
$$\frac{\text{area}(d, p_0^2, p_3^1)}{\text{area}(p_0^2, p_1^2, p_2^2)} = r,$$

which yields the G^2 condition

$$r^2 = r_- r_+.$$

For a G^2 spline, we can place the control points p_2^1 and p_1^2 anywhere, and then the joint point $p_3^1 = p_0^2$ is determined by the G^2 condition.

20.2 Curvature Calculations

We want to discuss the calculations and the derivations involved with the establishing the conditions for a G^2 spline. First, we want to consider the curvature of a cubic Bezier curve at one of the endpoints. This can be determined geometrically from the control points.

Recall from calculus, the curvature of a curvature is given by the normal component of acceleration as

$$\kappa = \frac{\mathbf{a} \cdot \mathbf{N}}{\|\mathbf{v}\|^2}$$

where $\mathbf{a}$ is the acceleration d^2c/dt^2, $\mathbf{N}$ is the unit normal vector, and $\mathbf{v}$ is the velocity vector. Our definition of curvature of a plane curve agrees with this definition except that we have a different definition of a normal vector vector of a plane curve.

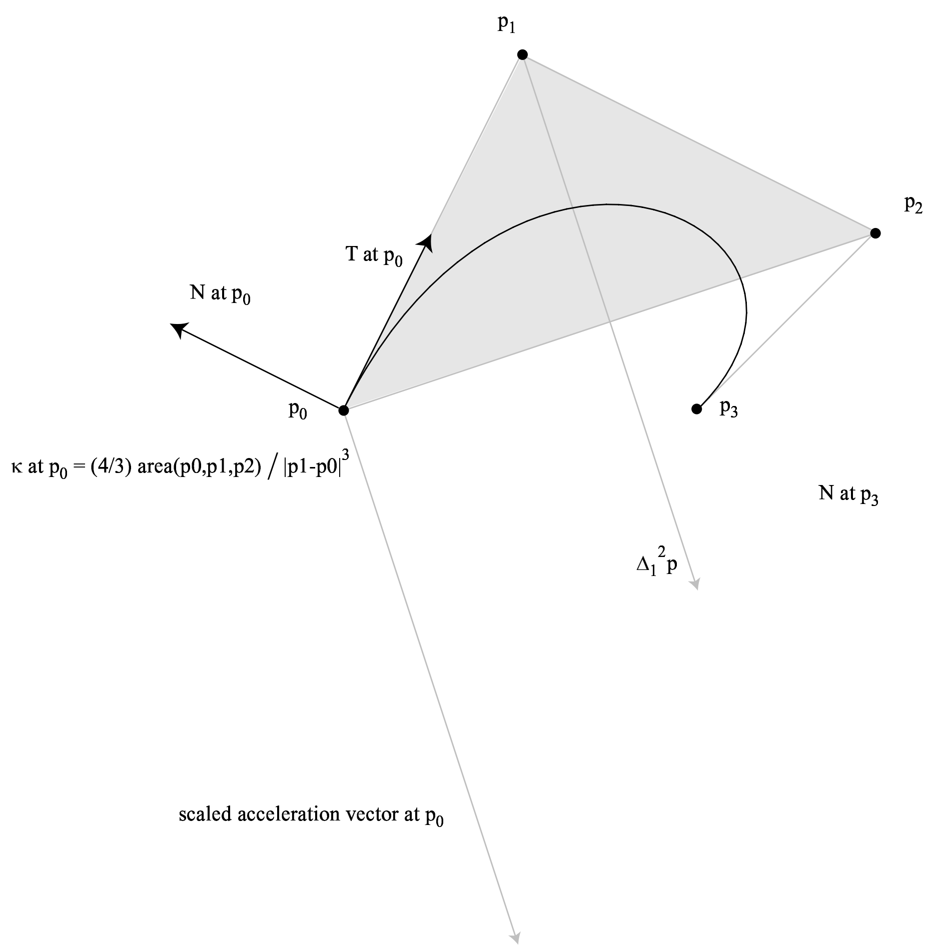

Figure 1: Curvature calculation for a cubic Bezier curve at p_0

For a cubic Bezier curve with control points p_0, p_1, p_2, p_3, the acceleration vector at p_0 is $6\Delta_0^2 p$ and the velocity vector is $3\Delta_0^1 p$, where $\Delta_0^2 p = (p_2 - p_1) - (p_1 - p_0)$ and $\Delta_0^1 p = p_1 - p_0$.

We can not write the unit normal vector N at p_0 in terms of the control points, but that is unneeded. It is important that the dotproduct of $\Delta_0^2 p$ with N is equal to the dotproduct of $p_2 - p_1$ with N, because $p_1 - p_0$ is parallel to T and therefore orthogonal to N. Thus, the area of the triangle with vertices p_0, p_1, p_2 is equal to

$$\text{area}(p_0p_1p_2) = \tfrac{1}{2}|(p_2 - p_1) \cdot N|\, \|p_1 - p_0\|$$

as $(p_2 - p_1) \cdot N$ is the height of the triangle $p_0p_1p_2$. We thus have that

$$a \cdot N = 6\Delta_0^2 p \cdot N = 6(p_2 - p_1) \cdot N = \frac{12\text{area}(p_0p_1p_2)}{\|p_1 - p_0\|}$$

and the curvature at p_0 is then

$$\kappa = \frac{12\text{area}(p_0p_1p_2)}{9\|p_1 - p_0\|^3} = \frac{4}{3}\,\frac{\text{area}(p_0p_1p_2)}{\|p_1 - p_0\|^3}$$

since $v = 3(p_1 - p_0)$. Likewise the curvature at p_3 is given by

$$\kappa = \frac{4}{3}\,\frac{\text{area}(p_1p_2p_3)}{\|p_3 - p_2\|^3}$$

which yields the two curvature formula that gave above. We note that these formula can only be used at the endpoints, since they were derived purely from the special form of the velocity and acceleration vectors at the endpoints.

20.3 Control Points for G^2 Splines

In creating a G^2 spline using two segments, we give five control points d_{-2}, d_{-1}, d_0, d_1 and d_2, and define a C^0 cubic Bezier spline, similar to the construction of a C^2 Bezier spline. The control points b_0, b_1, b_2, b_3 (the joint point), b_4, b_5 and b_6 of the C^0 Bezier spline are given by

$$b_0 = d_{-2}, \quad b_1 = d_{-1}, \quad b_5 = d_1, \quad b_6 = d_2$$

and then

$$b_2 = (1 - t_-)\, d_{-1} + t_-\, d_0 \quad b_4 = (1 - t_+)\, d_0 + t_+\, d_1$$

where $r_- = t_-/(1 - t_-)$ is a control specifying the location of b_2 on the line $d_{-1}d_0$ and $r_+ = t_+/(1 - t_+)$ is a control specifying the location of b_4 on the line d_0d_1. The quantities r_- and r_+ are the ratios of $\text{ratio}(d_{-1}, b_2, d_0)$ and $\text{ratio}(d_0, b_4, d_1)$; the ratio of three collinear points with the middle point B between the endpoints A and C is defined by $\text{ratio}(A, B, C) = |CB|/|AB|$. The joint point b_3 is then given by

$$b_3 = (1 - t)\, b_2 + t\, b_4$$

where $r = t/(1 - t)$ must satisfy $r^2 = r_- r_+$.

To show that $r^2 = r_- r_+$, we begin by noting that for d_{-2}, d_{-1}, d_0, d_1, d_2 to define a G^2 spline we must have

$$\frac{\text{area}(p_1p_2p_3)}{\|p_3 - p_2\|^3} = \frac{\text{area}(p_3p_4p_5)}{\|p_4 - p_3\|^3},$$

or

$$(*) \qquad \frac{\text{area}(p_1p_2p_3)}{\text{area}(p_3p_4p_5)} = \frac{\|p_3 - p_2\|^3}{\|p_4 - p_3\|^3}.$$

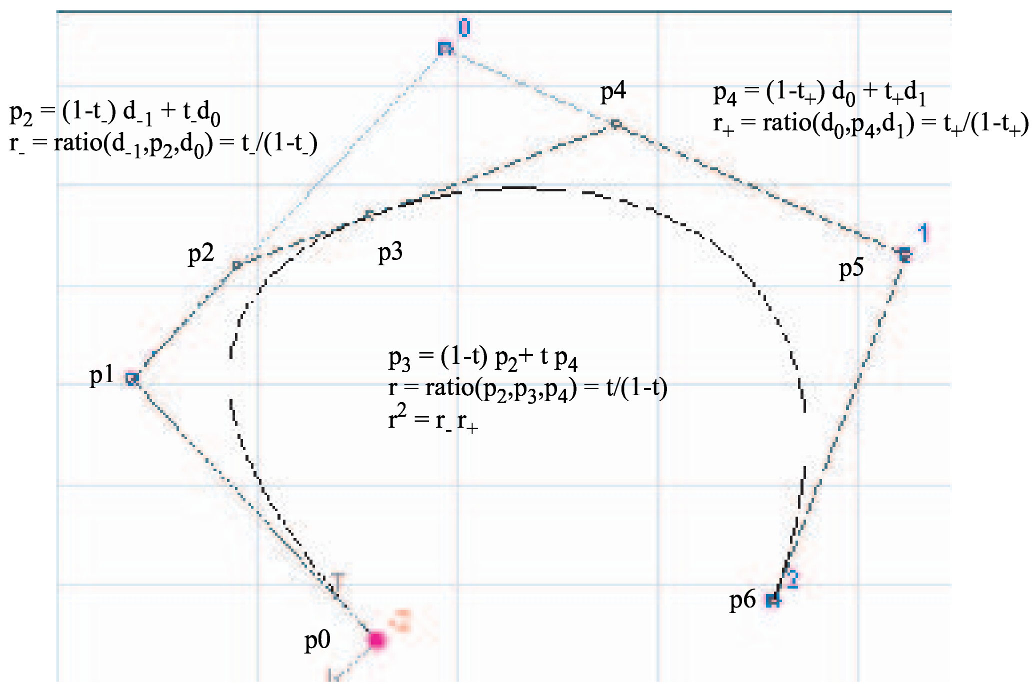

Figure 2: Determining the control points by ratios

The ratio of the areas in $(*)$ (the left-hand side of the equation) is given by

$$\frac{|(p_2 - p_1) \cdot N| \, \|p_3 - p_2\|}{|(p_5 - p_4) \cdot N| \, \|p_4 - p_3\|}$$

but $\|p_3 - p_2\| = t\|p_4 - p_2\|$ and $\|p_4 - p_3\| = (1-t)\,\|p_4 - p_2\|$. Thus the ratio of areas is equal to

$$r\,\frac{|(p_2 - p_1) \cdot N|}{|(p_5 - p_4) \cdot N|} = r\,\frac{t_-}{1 - t_+}\,\frac{|(d_0 - d_{-1}) \cdot N|}{|(d_0 - d_1) \cdot N|}$$

since $p_2 - p_1 = p_2 - d_{-1} = t_-\,(d_0 - d_{-1})$ and $p_5 - p_4 = (1 - t_+)\,(d_1 - d_0)$ To simplify further, we look at the triangle $p_2 d_0 p_4$, and note that $(d_0 - p_2) \cdot N = (d_0 - p_4) \cdot N$ (height is equal). Therefore, since $(d_0 - p_2) = (1 - t_-)(d_0 - d_{-1})$ and $d_0 - p_4 = t_+(d_0 - d_1)$, we get

$$\frac{(d_0 - p_2) \cdot N}{(d_0 - p_4) \cdot N} = \frac{(1 - t_-)}{t_+}\,\frac{(d_0 - d_{-1}) \cdot N}{(d_0 - d_1) \cdot N},$$

and thus

$$\frac{(d_0 - d_{-1}) \cdot N}{(d_0 - d_1) \cdot N} = \frac{t_+}{1 - t_-}.$$

Therefore, the ratio of areas in $(*)$ is

$$r\frac{t_-}{1 - t_+}\frac{t_+}{1 - t_-} = r r_- r_+$$

and the right-hand side of $(*)$ simplifies to r^3 which yields $r^2 = r_- r_+$.

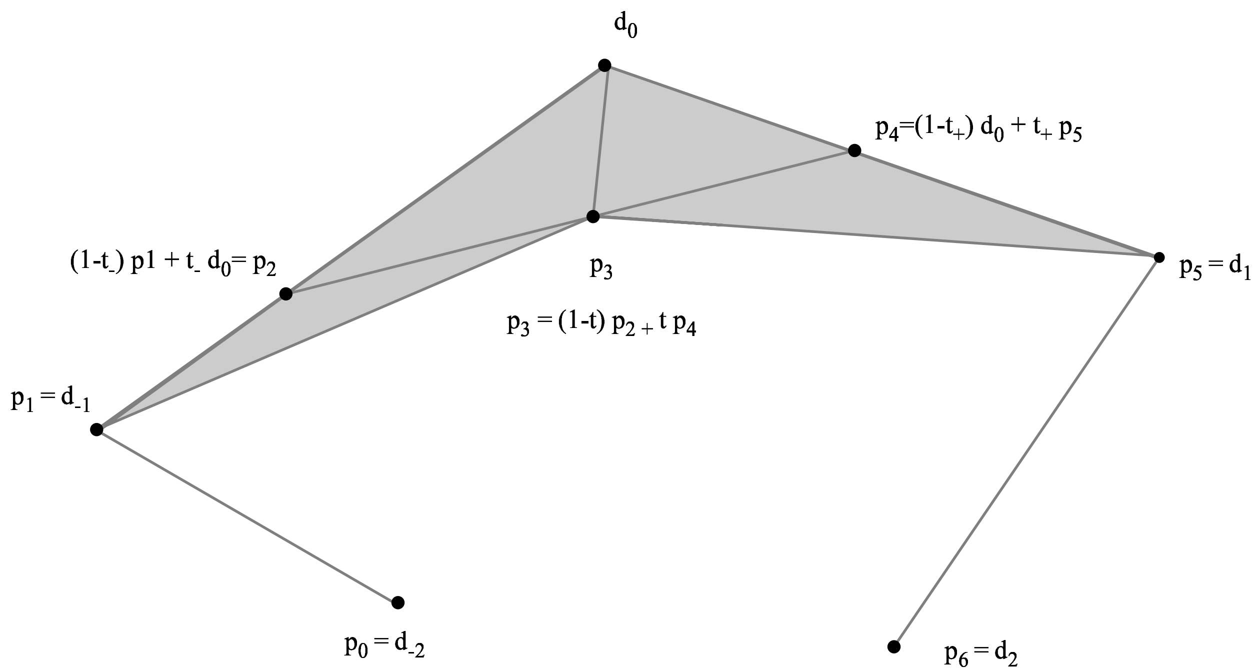

Figure 3: Determining the control points for G^2 Bezier spline

Exercises

1. For the cubic Bezier curve with control points $p_0 = [0, 0]$, $p_1 = [1, 1]$, $p_2 = [2, 1]$, $p_3 = [3, 1]$, compute the curvature at p_0 and p_3

   (a) Using calculus $\kappa = (a \cdot N)/|v|^2$ where v is the first derivative and a is the second derivative of the curve.

   (b) Using the formula $\kappa = \frac{4}{3}\,\text{area}(p_1 p_2 p_3)/\|p_3 - p_2\|^3$

2. Complete the interactive exercises associated with the applet G^2 Bezier splines.

3. For a C^0 Bezier spline of two segments with a non-uniform knot sequence determine the conditions for the curve to be joined in a C^2 manner, and then determine the location of the C^2 Bezier spline control points d_{-2}, d_{-1}, d_0, d_1 and d_2. Do you believe that this is the same construction as a G^2 Bezier spline of two cubic curves.