MA 323 Geometric Modelling
Course Notes: Day 21
Three Dimensional Bezier Curves,
Projections and
Rational Bezier Curves

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Over the next few days, we will be looking at extensions of Bezier curves and Bezier splines to three dimensions, and how these extensions when projected allow one to construct planar curves. In addition, we will look at an extension of Bezier curves and Bezier splines to B-Splines in the planar and the spatial case.

21.1 Curve Constructions in Three Dimensions

In extending the planar methods to spatial bases, the algorithms require no modifications, as the algorithms are defined abstractly in terms of affine combinations with no regard to the location of the point as two dimensional or three dimensional (or even higher dimensional). In fact, the constructions of Bezier curves and Bezier splines work exactly as before. All that we need to do is supply three dimensional points instead of two dimensional points. The construction of higher dimensional curves (in dimensions other than two or three) will not be explicitly mentioned further. We limit ourselves to two dimensional curves and three dimensional curves as our concern is constructing mathematical models of physical objects. The higher dimensional versions of curves occasionally are needed in geometric modelling to model the characteristics of an object, that is position of the object plus color of the object, texture of the object, density of the object, specular nature of the object. The properties of the object are considered as functions on the object and thus become other coordinates for the object. However, our concern in this course is only with the shape of the object.

We introduce no construction algorithms today, but rather introduce the mathematics of projections and the graphics pipeline. When we view an object on a computer screen, we are not looking at a three dimensional object. We are looking at a two-dimensional representation of a three-dimensional object. We are looking at a projection of the object. In some sense, this is the beginnings of understanding the geometry of vision. We will not devote extensive time to the study of projective geometry, as that is an entire course and we only require the beginnings of projective geometry to understand the projection of a object.

21.2 Projections of Three Dimensional Objects

We will consider two types of projection for three dimensional objects, parallel projections and perspective projections. These projections are the two most common methods for looking at three dimensional objects as two dimensional figures. In the remainder of these

introductory remarks, we consider general remarks about the process of projecting three-dimensional objects onto a two dimension plane.

A projection for our purposes is a continuous map P (function) from three dimensional space $\mathbb{R}^3$ to two dimensional plane $\mathbb{R}^2$. An immediate consequence of projecting three dimensional objects onto two dimensions is that multiple points can project to the same point. The map P (since it is continuous) can not be one-to-one.

It is worth noting that we do not project the entire three-dimensional space onto the entire two-dimensional plane. We only project a portion of three dimensional space onto a portion of the two dimensional plane. It is typical to consider projections in terms of an imaging system so that the portion of space that is projected is considered the viewable portion of space, and the portion of the plane that is projected upon is the image that is seen. In fact, it is also typical to view the two dimensional plane $\mathbb{R}^2$ as a plane in three dimensional space $\mathbb{R}^3$, and call it the image plane. In this view, one can consider the image plane as the canvas on which the object is painted, see diagram below.
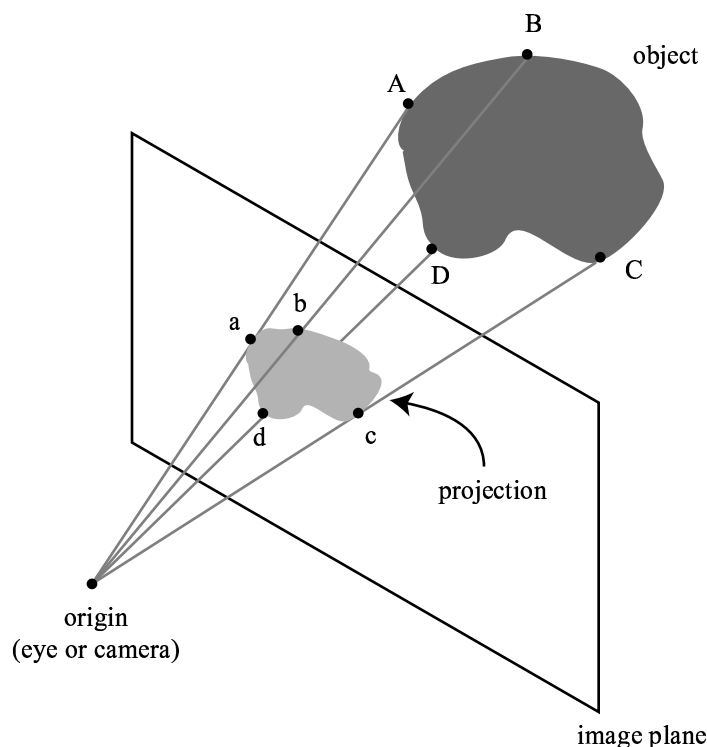

Figure 1: Perspective Projection, Image Plane and Camera

The projections that we are considering (parallel projections and perspective projections) involve (light) rays travelling through the object and intersecting the image plane, see diagram above. The use of rays (straight lines) as generating the projection greatly simplifies the analysis, and allows very simple geometrical analysis. In the parallel projection, the rays are assumed to all intersect the image plane perpendicularly, and in perspective projections the rays are assumed to all intersect at a point (considered either as a light source or a light sink or an eye).

21.3 Perspective Projection

The first type of projection that we want to consider is a perspective projection. This is the projection used by cameras, especially pin-hole cameras. In this projection, we view the origin of a coordinate system as the observers eye, and consider the image plane as the plane $x = 1$ with the viewable space as the half-space $x > 0$. The applets for this course restrict the position so that you can only place points with $x > 0$. There is no theoretical restriction that $x > 0$, and even that $x \neq 0$. One of the exercises asks you to look at the effect of $x < 0$ or $x = 0$ on a Bezier curve, (see rational Bezier curves). Perspective projection is defined by taking the set of lines that intersect at the origin and equating these lines with their intersection with the plane $x = 1$, see diagram below.
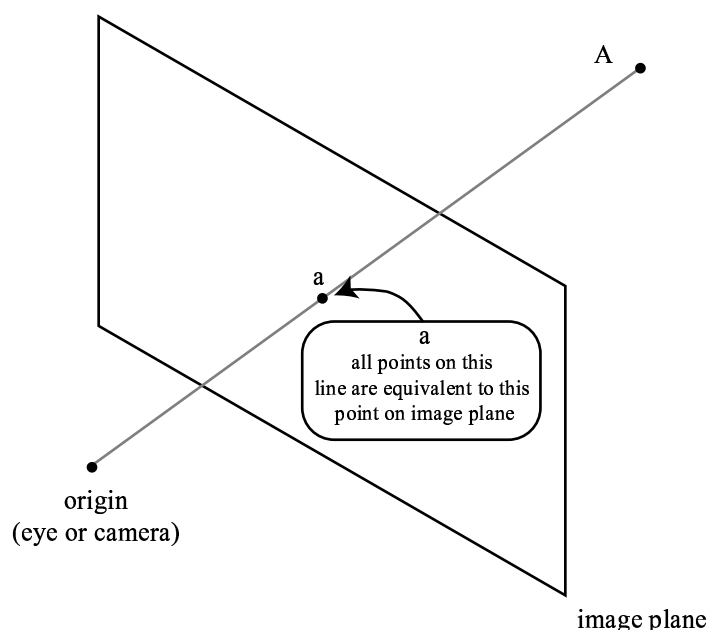

Figure 2: Perspective equivalence

Geometrically, it is useful to consider the projection P from $\mathbb{R}^3 \backslash \{\mathbf{0}\}$ (all of space except the orign) to the plane $x = 1$. With this view, lines that do not intersect the origin project to lines on the image plane. Let us consider the image of a line L in $\mathbb{R}^3$ under this projection. The point of intersection of the line L with the plane $x = 0$ (if it exists) is projected to a point at infinity. Given that the line intersects the plane $x = 0$, the portion of the line with $x > 1$ projects to a finite line segment. The endpoint corresponding to $x = \pm\infty$ is called a vanishing point of the line. The line can then be expressed as $L = [t, y_0 + y_1 t, z_0 + z_1 t]$. The point (y_0, z_0) is the intersection of the line L with the plane $x = 0$ and the point (y_1, z_1) is the vanishing point of the line. The portion of the line with $x < 1$ is then consists of two disconnected rays.

Perspective projections have a different feel, and are unlike parallel projections which are used more commonly for displaying objects. For instance, consider the cubes in the diagram below. The larger square is in the foreground (it is closer to the viewer, and is the front face of the cube), and the smaller square is parallel to the larger square and the same size (it is the back face of the cube, and farther from the viewer). In this view the front and back face of the cube are parallel to the image plane, and thus the faces appear as cubes. The other

q_0

q_0

q

$p_1=q_1$
intersection of line
and plane x=1

q_+

q_+

q_+
projection of point at infinity

p

p_0
intersection of line
and plane x=0

origin
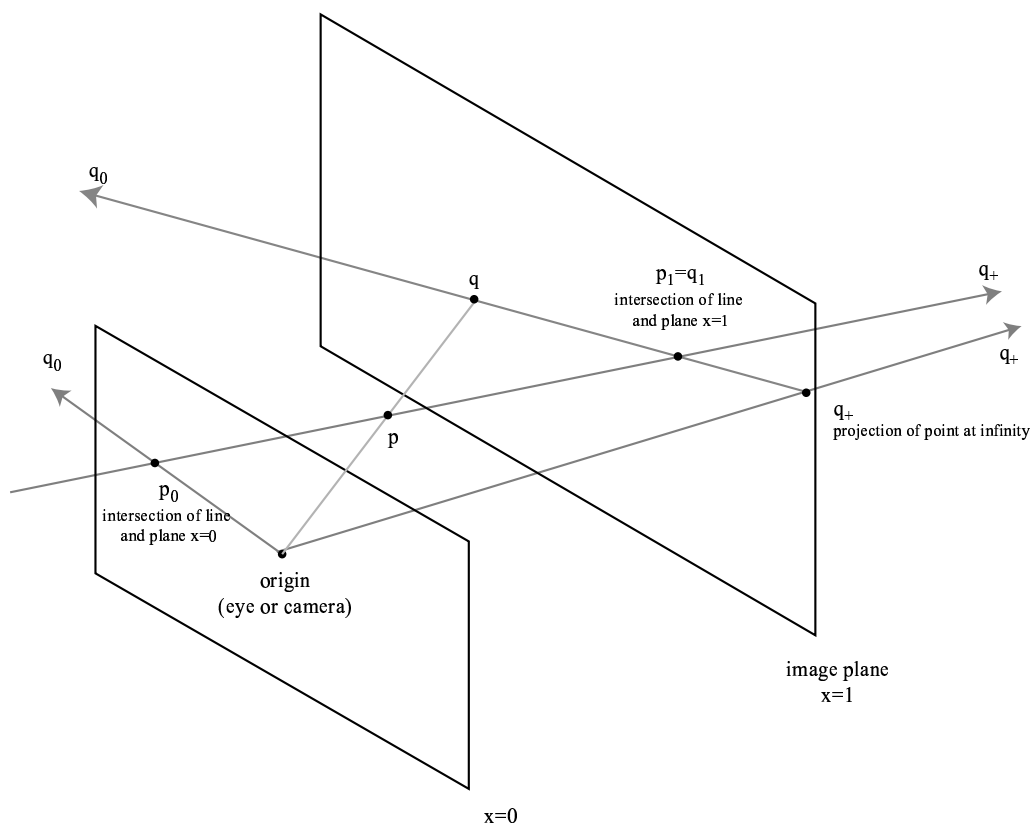(eye or camera)

image plane
x=1

x=0

Figure 3: Perspective projections of lines

two pairs of parallel faces (top and bottom faces and the left and right faces) have parallel edges, and thus the extended lines share a common vanishing point (proved below).

As illustrated in the above projections of a cube, one difference between a perspective projection and parallel projections is that parallel lines can project to non-parallel lines. For instance, the lines $L = [-t, y_0 - y_1 t, z_0 - z_1 t]$ and $L' = [-t, \eta_0 - y_1, \zeta_0 - z_1 t]$ are parallel to the image plane $x = 1$ intersecting at the plane $x = 0$ at points (y_0, z_0) and (η_0, ζ_0) but having the same vanishing point on the image plane (y_1, z_1). In fact, lines that are parallel (and not contained in a plane $x = constant$) will intersect at the same vanishing point. This is simple result of homogeneous coordinates, see below. For a perspective projection of a cube, there are three important vanishing points; the vanishing points of the three sets of parallel lines that form a cube. It is important to note that these points are possible at infinity if the lines are parallel to the image plane itself.

Another difference between perspective projections and parallel projections is that distances are not preserved. This should be immediately obvious after thinking about the effect of parallel lines intersecting at the same vanishing point. An infinitely long line segment projects to a finite line segment. Different parallel lines will intersect at the same point, so they have different slopes and thus will have different lengths. The calculation of lengths also becomes more apparent with the introduction of homogeneous coordinates.
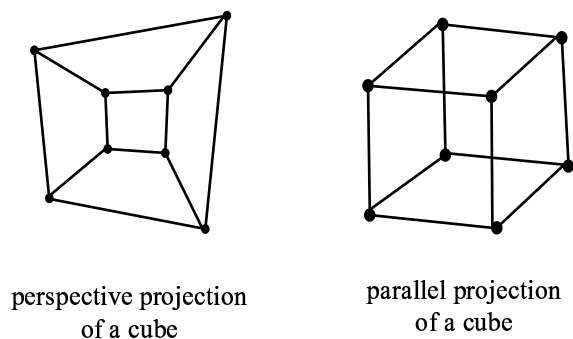
perspective projection
of a cube

parallel projection
of a cube

Figure 4: Difference between parallel and perspective projections of the same cube

21.4 Parallel Projections

Parallel projections involve having parallel lines in space project to parallel lines in the plane, and parallel line segments with equal length projecting to parallel lines with equal length. The projected lines will in general not have the same distance as the original lines in three dimensions, but they will have a common ratio to the projected line. Most engineering drawings are produced using parallel projections. The three dimensional plots obtained in Maple, Mathematica, MATLAB all use parallel projections.

To explain parallel projections, it is easiest to give one and then explain how it reflects a parallel projection. Consider the drawing below of the cubes below. Each face of the cube is represented by a parallelogram with equal length sides (a rhombus) not a square. The angles in the parallelograms not being right angles give an indication of the position from which you are viewing the cube. The equality of the lengths is meant to indicate that it is a cube (or at least a rectangular solid). The exact scale is implied, and viewed relatively.
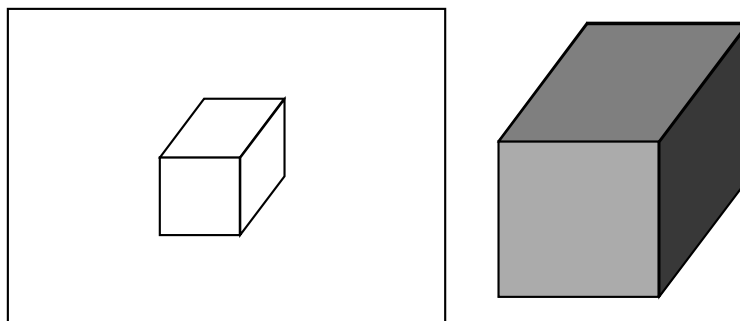

Figure 5: A parallel projection of a cube

Typically, one needs a test cube in order to orient one-self and understand a parallel projection. There are two typical parallel projections that we will consider; the military projection, where the side lengths of the test cube are equal and the cavalier projection where the front/back faces of the test cube are squares and the left/right faces are obtained using a forty-five degree angle and side length of 1/2, see diagrams below.

In the three-dimensional applets associated with this course, you can view the objects created with respect to either a perspective projection or a parallel projection, see screen shots on
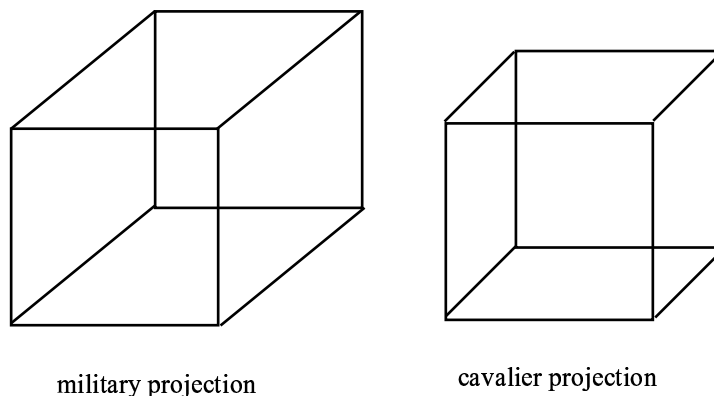
military projection   cavalier projection

Figure 6: Test Cubes for Military and Cavalier Projections

the next page. You are allowed only to manipulate the curve in the perspective projection, as you are drawing on the image plane but can set the perspective location
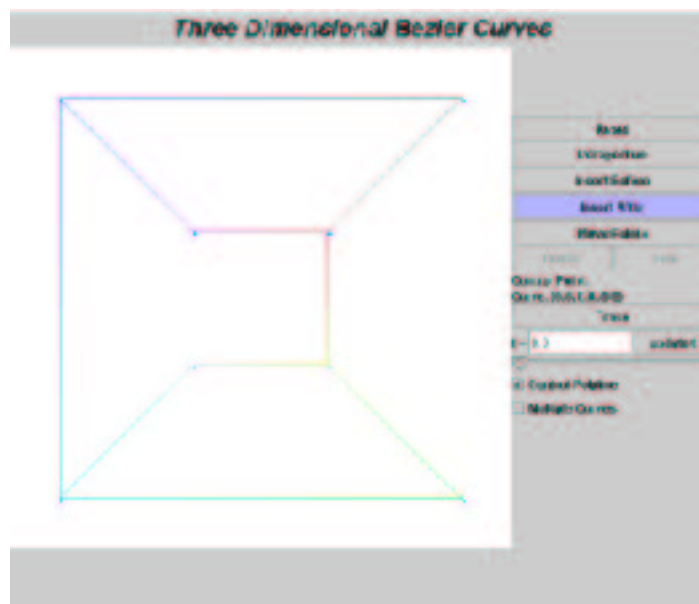

Figure 7: Perspective Projection for Applets

The method for creating a parallel projection is to take a linear map (a matrix) A from $\mathbf{R}^3$ to $\mathbf{R}^2$ and look at the the affine map $\mathbf{y} = \mathbf{p} + A\mathbf{x}$ for $\mathbf{x} \in \mathbf{R}^3$ and $\mathbf{y}, \mathbf{p} \in \mathbf{R}^2$. The cavalier projection is given by

$$y_1 = x_2 + (x_1/2)\sqrt{2}/2$$
$$y_2 = x_3 + (x_1/2)\sqrt{2}/2$$

with $\mathbf{y} = (y_1, y_2)$ and $\mathbf{x} = (x_1, x_2, x_3)$. In this situation, we are working with a different convention than normal, see diagram on the next page. This is the convention used in the applets, which is slightly different than the normal mathematical conventions. As the applets use a variation of the cavalier projection as the parallel view.
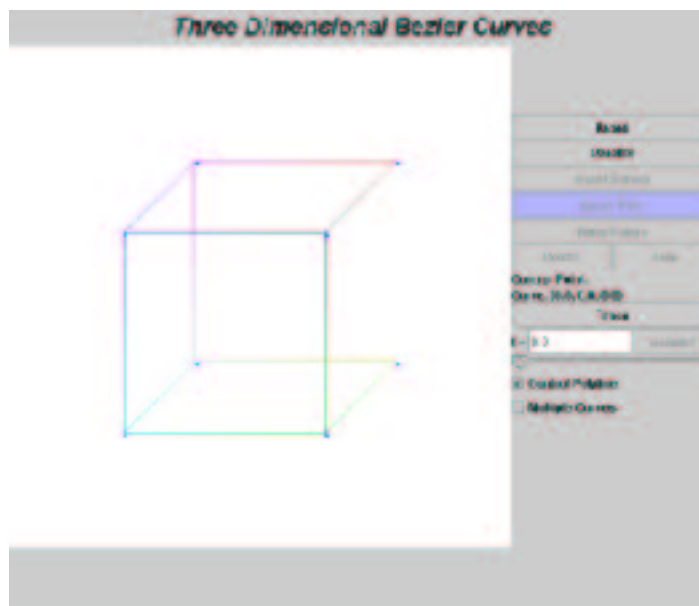
Figure 8: Parallel Projection for Applets

21.5   Homogeneous Coordinates

In constructing a perspective projection, it is standard to work with coordinates. The coordinates on the image plane are *homogeneous coordinates*. Given a point (x, y, z) in $\mathbb{R}^3$, the projection on the image plane is then $(y/x, z/x)$. This is because the line through the point (x, y, z) is (tx, ty, tz). Solving for $tx = 1$ (the image plane), we have $t = 1/x$ and thus the point on the image plane is $(1, y/x, z/x)$. This makes the perspective calculation easy, just divide by the x coordinate.

Homogeneous coordinates and calculations with homogeneous coordinates imply that parallel lines have the same intersection point. Two lines in space are parallel if they have parallel direction vectors. Recall, a line in space is given as $\mathbf{p} + t\mathbf{v}$ where $\mathbf{p} = (p_1, p_2, p_3)$ is a point on the line and $\mathbf{v} = (v_1, v_2, v_3)$ is the direction vector. The image of the point $\mathbf{p} + t\mathbf{v}$ is the point

$$\left(\frac{p_2 + v_2 t}{p_1 + v_1 t}, \frac{p_3 + v_3 t}{p_1 + v_1 t} \right).$$

As $t \to \pm\infty$, the image point has the limiting value $(v_2/v_1, v_3/v_1)$ for the *vanishing point*. Notice the vanishing point depends only on the direction vector, and thus is the same for all lines parallel to the same vector. The length of the direction vector is immaterial, because we are considering a ratio of the components of the direction vector.

It is also easy to see that two parallel line segments with the same length will project to different lengths. In this calculation, the points on the line are important. Let $\mathbf{p}$ and $\mathbf{q}$ be two distinct point in the plane $x = 1$, and consider the parallel line segments $\mathbf{p}, \mathbf{p} + \mathbf{v}$ and $\mathbf{q}, \mathbf{q} + \mathbf{v}$ for some vector $\mathbf{v} = (v_1, v_2, v_3)$ with $v_1 > 0$. Under a perspective projection, we have the lengths of the projected lines as

$$l_1 = \left\| \left(\frac{p_2 + v_2}{p_1 + v_1} - \frac{p_2}{p_1}, \frac{p_3 + v_3}{p_1 + v_1} - \frac{p_3}{p_1} \right) \right\| \quad \text{and} \quad l_2 = \left\| \left(\frac{q_2 + v_2}{q_1 + v_1} - \frac{q_2}{q_1}, \frac{q_3 + v_3}{q_1 + v_1} - \frac{q_3}{q_1} \right) \right\|$$

which simplifies to

$$l_1 = \frac{\sqrt{(v_2 - p_2 v_1)^2 + (v_3 - p_3 v_1)^2}}{1 + v_1} \quad \text{and} \quad l_2 = \frac{\sqrt{(v_2 - q_2 v_1)^2 + (v_3 - q_3 v_1)^2}}{1 + v_1}$$

using $p_1 = 1$ and $q_1 = 1$. Notice that the distance depends on the points $\mathbf{p}$ and $\mathbf{q}$ chosen and the vector $\mathbf{v}$. For instance, with $\mathbf{p} = (1, 1, 0)$ and $\mathbf{q} = (1, 0, 1)$ we have $l_1 = \sqrt{(v_2 - v_1)^2 + v_3^2}/(1 + v_1)$ and $l_2 = \sqrt{v_2^2 + (v_3 - v_1)^2}/(1 + v_1)$ which will not in general be equal.

21.6   Bezier Curves and Bezier Splines in Space

In creating Bezier curves and Bezier splines in space, the algorithms work the same. A Bezier curve is constructed used de Casteljau's algorithm or using the Bernstein polynomials. Bezier splines are constructed according to the algorithms provided in the previous chapter, where the control points for a piecewise Bezier curve are constructed according to the rules for joining the individual cubic Bezier curves in the appropriate manner.

However, you will see differences in the image produced depending on the projection used to display the curve. In a parallel projection, you will see no difference in the algorithm. This is because parallel projections are affinely invariant, meaning the barycentric combinations are preserved under parallel projections. Perspective projections are not affinely invariant, affine combinations are not preserved under perspective projections. This is due to the distances not being preserved under a perspective projection. The joint points for a C^1 spline will not appear to always be the midpoint of the line segment joining the two control points.

It is important to notice that all of the properties for Bezier curves still hold for spatial Bezier curves. There are a few differences though. The convex hull is no longer a planar object but a spatial object, defined in the exact same way. In the variation diminishing property, instead of planes instead of lines to define the property, as the number of intersections of the curve and a plane is less than (or equal to) the number of intersection of the control polyline and the the plane.

21.7   Rational Bezier Curves and Rational Bezier Splines

The perspective projection of a spatial Bezier curve yields a rational Bezier curve,

$$\xi(t) = y(t)/x(t) \quad \text{and} \quad \eta(t) = z(t)/x(t),$$

as a rational curve is defined to be curve where each component function is a rational function or a ratio of polynomials. or

$$\xi(t) = \frac{\sum_{i=0}^{n} \beta_i^n(t) \, y_i}{\sum_{i=0}^{n} \beta_i^n(t) x_i}$$

$$\eta(t) = \frac{\sum_{i=0}^{n} \beta_i^n(t) \, z_i}{\sum_{i=0}^{n} \beta_i^n(t) x_i}$$

where the control points in space are $p_i = (x_i, y_i, z_i)$. Using the perspective equivalence of (x, y, z) and $(1, y/x, z/x)$, one can view the point $\bar{p} = (y/z, z/x)$ as a point in the image plane and $\lambda = x$ as the *weight* of $\bar{p}$. This means that a rational Bezier curve is described by

$$c(t) = \frac{\sum_{i=0}^{n} \beta_i^n(t) \, \lambda_i \bar{p}_i}{\sum_{i=0}^{n} \beta_i^n(t) \, \lambda_i}$$

This is an affine combination on the image plane control points $\{\bar{p}_i\}$, as it can be put in the form

$$c(t) = \sum_{i=0}^{n} \alpha_i^n(t)\, \bar{p}_i$$

where

$$\alpha_i^n(t) = \frac{\beta_i^n(t)\, \lambda_i}{\sum_{i=0}^{n} \beta_i^n(t)\, \lambda_i}.$$

The basis functions $\alpha_i^n(t)$ sum to one, as a simple consequence of their definition,

$$
\begin{aligned}
\sum_{i=0}^{n} \alpha_i^n(t) &= \sum_{i=0}^{n} \frac{\beta_i^n(t)\, \lambda_i}{\sum_{i=0}^{n} \beta_i^n(t)\, \lambda_i} \\
&= \frac{\sum_{i=0}^{n} \beta_i^n(t)\, \lambda_i}{\sum_{i=0}^{n} \beta_i^n(t)\, \lambda_i} \\
&= 1
\end{aligned}
$$

using the common denominator of all the basis functions $\alpha_i^n(t)$.

The majority of properties of Bezier curves still hold for rational Bezier curves. In particular, rational Bezier curves have end-point interpolation, prescribed tangent lines at the end-points, convex hull property (assuming the weights are positive), and variation diminishing property. The symmetry property (that the same curve is obtained by reversing the order of the sequence of control points) still holds also, but it is important to understand that the weights λ_i are associated to the projected points $\bar{p}_i$.

Rational Bezier splines are created as spatial Bezier splines and then projected perspectively to a planar curve. It is important to note that in the perspective projection, lengths are not preserved so that the placement of the joint points on the image plane are not determined via the same method as in the planar case. The weights λ_i play a role in the placement of the joint points. In fact, one way to view the construction of a rational curve (without using the perspective projection) is to define the construction abstractly through the weights. Viewing the weights as strictly positive numbers, a higher weight means the curve gets closer to the point and smaller weights mean the curve gets from the control point. The exact nature of the combination can be viewed as an analogue of a physical calculation; calculating the center of mass of a discrete collection of particle with varying weights. The point $c(t)$ on a rational curve is the center of mass of the points $\{\bar{p}_i\}$ with weights $\beta_i^n(t)\, \lambda_i$. Notice that the calculation is directly the same calculation. Therefore, the placement of the control points for the cubic rational curve associated to a spline depends highly on the weights of the points.

Exercises

1. Play with the applets for creating three dimensional curves. Notice the difference between images of the curves produced under parallel and perspective projections.

2. Can you create a smooth closed three dimensional curve (meaning the curve is C^1 or C^2 at the endpoints with $p_0 = p_n$? How, can you achieve this? What conditions need to be placed on the curve?

3. THOUGHT PROBLEM: For a rational Bezier curve (perspective projection of a spatial Bezier curve), what is the effect of $x < 0$ on the curve. For instance, what is the perspective projection of the spatial Bezier curve c_j with control points $x_0(1, 0, 0)$, $x_1(1, 1, 1)$, $x_2(1, 2, 0)$ and $x_3(1, 0, 3)$ with $x_j < 0$ and $x_i > 0$ for $i \neq j$, and how does it compare to planar Bezier curve with control points $(0, 0)$, $(1, 1)$, $(2, 0)$ and $(0, 3)$, and the convex hull of the control polyline of this planar Bezier curve.

4. Show that the projection (parallel or perspective) of a convex object in space is a convex object in the plane. This shows that a rational Bezier curve has the convex hull property directly from the fact that a Bezier curve has the convex hull property (assuming positive weights, x-coordinates, and using the image plane control points $\bar{p} = (y/x, z/x)$.