MA 323 Geometric Modelling
Course Notes: Day 18
Bezier Splines II

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Today, we want to continue our discussion of Bezier splines concentrating on creating C^2 Bezier splines, twice differentiable piecewise defined curve from cubic Bezier curves. Constructing a twice differentiable piecewise defined curve from cubic Bezier curves that has the same properties as Bezier curves (endpoint interpolation, prescribed tangent lines at the endpoints, symmetry, convex hull property and variation diminishing) and has the local control of splines (moving one control point only affects a portion of the curve) requires significantly more work that the extension from C^0 splines to C^1 splines.

The first question that should be considered is what is the advantage of looking at C^2 curves. For motion problems (animation or motion planning problems), C^2 curves are necessary to ensure the acceleration is continuous, a desirable property. One normally wants the acceleration bounded and continuous with bounds on the derivative of the acceleration. For designing parts, C^2 curves ensure the curvature is continuous which is useful in geometrically understanding the curve. The exact nature of curvature in describing a curve will be discussed tomorrow.

18.1 C^2 cubic Bezier splines

We saw in our discussion of Hermite curves that one can create a C^2 curve from quintic curves. We defined quintic Hermite curve by specifying the first and second derivatives at the *joint points*. However, as we saw controlling the shape of the curve in this fashion is not entirely intuitive. In this subsection, we will look at how to join two (or more) cubic curves together in a C^2 fashion. This is possible because we will determine the location of the joint point instead of specifying its location as in Hermite curves. We will begin by considering the joining of two cubic curves, and then extend to consider the joining of more curves.

Given control points for two cubic Bezier curves $p_0^1, p_1^1, p_2^1, p_3^1$ and $p_0^2, p_1^2, p_2^2, p_3^2$. We first need $p_3^1 = p_0^2$ so the curve is continuous, and next we need $p_3^1 - p_2^1 = p_1^2 - p_0^2$ or $p_3^1 = p_0^2 = (p_1^2 + p_2^1)/2$ so the piecewise curve is differentiable (the derivative at the joint must be equal). The new condition so the curve is C^2 is the equality of the second derivative at the joint point. The equality of the second derivative is equivalent to the equality a condition on the equality of the second differences. In other words, $p_3^1 - 2p_2^1 + p_1^1 = p_2^2 - 2p_1^2 + p_0^2$. Manipulating this equation, we have

$$
\begin{aligned}
(p_3^1 - p_2^1) - (p_2^1 - p_1^1) &= (p_2^2 - p_1^2) - (p_1^2 - p_0^2) \\
(p_3^1 - p_2^1) + (p_1^2 - p_0^2) &= (p_2^1 - p_1^1) + (p_2^2 - p_1^2) \\
2(p_3^1 - p_2^1) = 2(p_1^2 - p_0^2) &= (p_2^1 - p_1^1) + (p_2^2 - p_1^2)
\end{aligned}
$$

To understand the manipulation above, we look at in terms of a vector equation, which can be best understood through the diagram below.

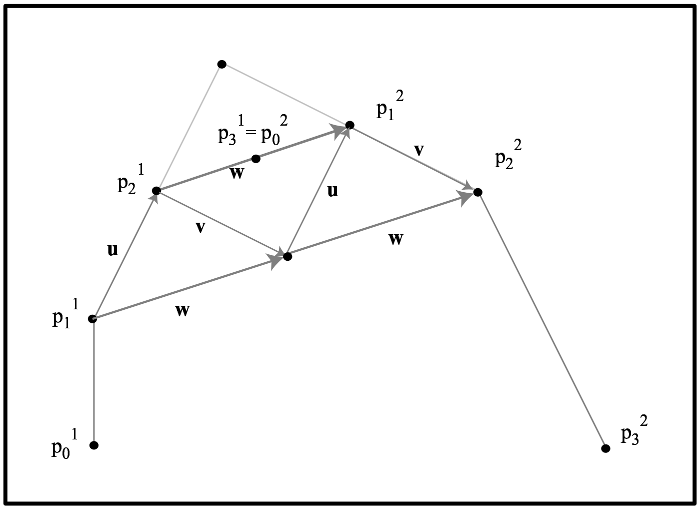

Figure 1: A vector view of the equality of second differences

The diagram above implies that the segment S_1 formed by p_2^1 and p_1^2 is parallel to the segment S_2 formed by p_1^1 and p_2^2. Moreover, the length of S_2 is twice that of S_1. Therefore, the lines l_1 and l_2 through p_1^1, p_2^1 and p_1^2, p_2^2 intersect at a point X. The triangles $p_1^1 X p_2^1$ and $p_2^1 X p_2^2$ are similar. Thus p_2^1 is the midpoint of the segment formed by p_1^1 and X and p_1^2 is the midpoint of the segment formed by p_2^2 and X. (See diagram below)

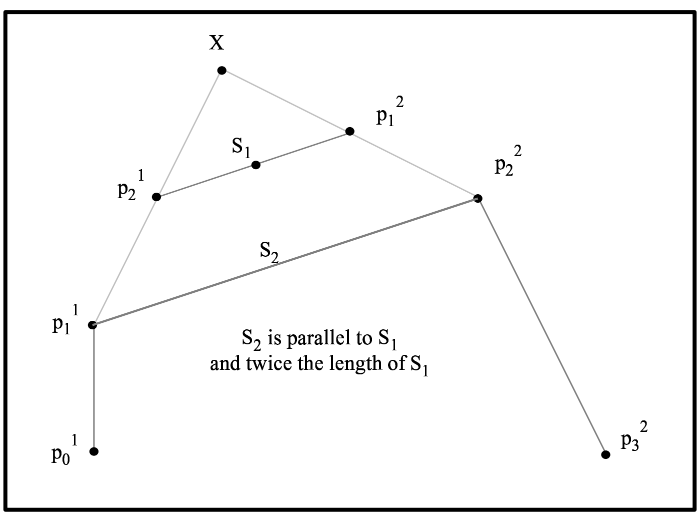

Figure 2: Geometric Properties of Equality of Second Differences

Therefore, to create a C^2 curve with two cubic curves, one needs only the five points p_0^1, p_1^1,

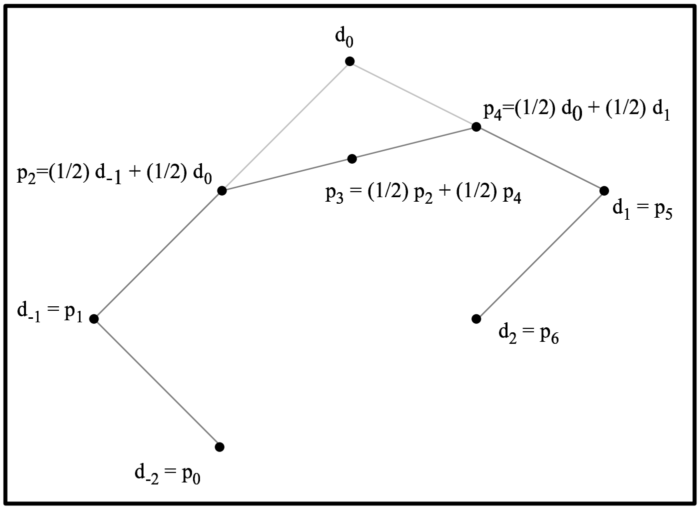

Figure 3: Constructing a C^2 Bezier spline of two cubic segments

X, p_2^2 and p_3^2. This is because

$$
\begin{aligned}
p_2^1 &= \frac{1}{2}p_1^1 + \frac{1}{2}X \\
p_1^2 &= \frac{1}{2}X + \frac{1}{2}p_2^1 \\
p_3^1 = p_0^2 &= \frac{1}{2}p_2^1 + \frac{1}{2}p_1^2.
\end{aligned}
$$

However, it is more convenient to use other notation. But these are the correct points to provide, so that the properties of Bezier curves (endpoint interpolation, prescribed tangent lines, symmetry, convex hull property and variation diminishing property) are preserved in our construction of C^2 Bezier curve.

For creating a C^2 cubic Bezier spline with two segments, one gives 5 points d_{-2}, d_{-1}, d_0, d_1, d_2 and defines the control points p_0, p_1, p_2, $\cdots$, p_6 for a C^0 Bezier spline of two segments as follows. Define

$$p_0 = d_{-2} \quad \text{and} \quad p_6 = d_2 p_1 = d_{-1} \qquad \text{and} \quad p_5 = d_1$$

then

$$p_2 = \frac{1}{2}d_{-1} + \frac{1}{2}d_0 \quad \text{and} \quad p_4 = \frac{1}{2}d_0 + \frac{1}{2}d_1$$

and finally

$$p_3 = \frac{1}{2}p_2 + \frac{1}{2}p_3.$$

Notice the symmetry of the algorithm, so reversing the order of the spline control points d_i we get the same curve. We also note that we have interpolation of the first and last control points, and the tangent lines at the first and last control points are given by the lines $d_{-2}d_{-1}$ and $d_1 d_2$.

To extend this beyond two cubic segments requires a different point of view. The construction is slightly more complicated. We specify the spline by $L + 3$ spline control points, d_i with $i = -2, -1, 0, 1, \cdots, L$. This is again an improvement over C^0 and C^1 splines in terms of the number of control points needed, as each additional segment requires one additional point. In the algorithm given below, we let p_i be the control points for the piecewise cubic Bezier curve that makes the spline. Recall, each individual cubic curve in the spline is defined using the control points p_{3i-3}, p_{3i-2}, p_{3i-1} and p_{3i} where $i = 1, 2, \cdots, L$ with L the number of cubic curves in the segment. The reason for starting the index for d_i at -2 is purely so the last index is L, and determines the number of curve segments.

For a C^2 cubic spline with 3 segments, things are slightly more complicated. The First segment is defined by d_{-2}, d_{-1}, d_0, and d_1, the middle curve is defined by the points d_{-1}, d_0, d_1 and d_2, and the last segment is defined by d_0, d_1, d_2, and d_3. The trick is getting two control points out of the segment d_0 and d_1. This is accomplished by dividing the segment into equal thirds, to get the control points b_4 and b_5. The other control points are defined similar to for a spline with 2 segments. First, we set

$$p_0 = d_{-2}, \qquad p_1 = d_{-1}, \qquad p_8 = d_2, \qquad p_9 = d_3.$$

Next, we define

$$p_2 = \tfrac{1}{2}d_{-1} + \tfrac{1}{2}d_0, \qquad p_7 = \tfrac{1}{2}d_1 + \tfrac{1}{2}d_2.$$

Then, we define

$$p_4 = \tfrac{2}{3}d_0 + \tfrac{1}{3}d_1, \qquad p_5 = \tfrac{1}{3}d_0 + \tfrac{2}{3}d_1.$$

Finally, we set

$$p_3 = \tfrac{1}{2}p_2 + \tfrac{1}{2}p_4, \qquad p_6 = \tfrac{1}{2}p_5 + \tfrac{1}{2}p_7.$$

See diagram below

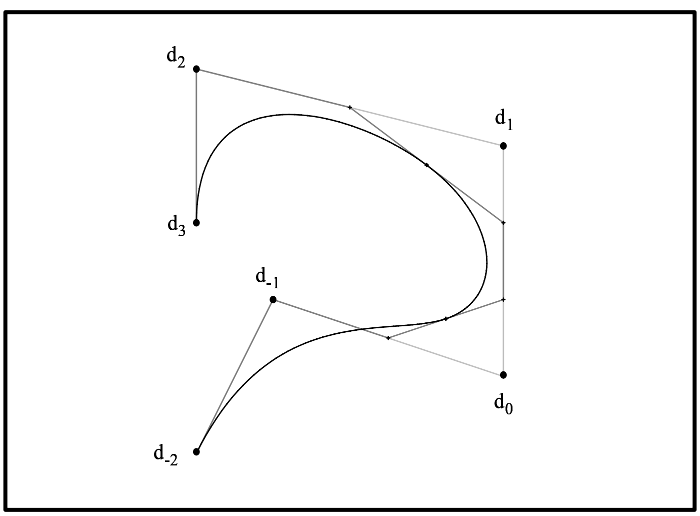

Figure 4: A C^2 Bezier spline of three cubic segments

In general, given control points d_{-2}, d_{-1}, d_0, d_1, $\cdots$, d_L, we define the piecewise Bezier control points as follows. First, we set

$$p_0 = d_{-2}, \qquad p_1 = d_{-1}, \qquad p_{3L-1} = d_{L-1}, \qquad p_{3L} = d_L.$$

Next, we define

$$p_2 = \tfrac{1}{2}d_{-1} + \tfrac{1}{2}d_0, \qquad p_{3L-2} = \tfrac{1}{2}d_{L-2} + \tfrac{1}{2}d_{L-1}.$$

Then, we define for $i = 1, 2, \cdots, L-2$ the points

$$p_{3i+1} = \tfrac{2}{3}d_{i-1} + \tfrac{1}{3}d_i \quad p_{3i+2} = \tfrac{1}{3}d_{i-1} + \tfrac{2}{3}d_i.$$

Finally, we set for $i = 1, 2, \cdots, L-1$ the points

$$p_{3i} = \tfrac{1}{2}p_{3i-1} + \tfrac{1}{2}p_{3i+1}.$$

See diagram below.

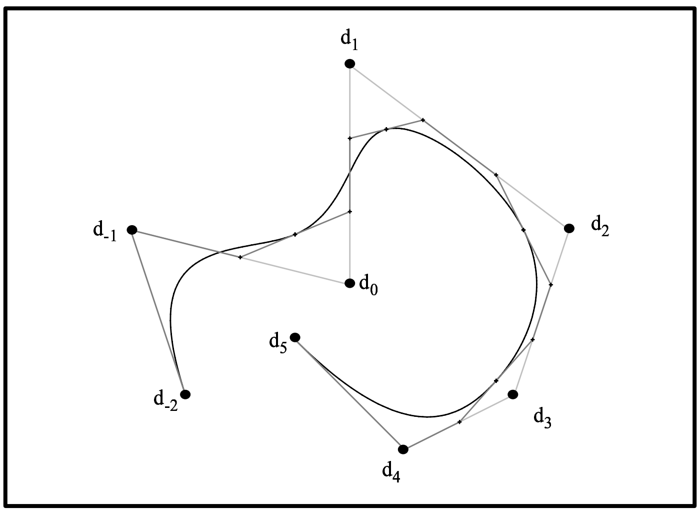

Figure 5: A C^2 Bezier spline of five cubic segments

Exercises

1. Play with the applet: Creating C^2 (cubic) Bezier Splines with a uniform knot sequence. Notice the difference between a C^1 Bezier spline and the Bezier curve with the same control points. In particular, play with how the control points affect the shape of the curve.

2. Determine the conditions required on the control points to create a closed C^2 Bezier spline.

3. Use the above conditions to create a C^2 approximation to a circle.

4. Determine the conditions needed to define a C^2 cubic Bezier spline using a nonuniform knot sequence.