MA 323 Geometric Modelling
Course Notes: Day 38
Triangulations and Creating Polyhedral Surfaces

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Today, we want to discuss how to construct the base polyhedron for starting the process for creating a subdivision surface. This is a basic problem. Up until now, we have concentrated on the process of creating the end surface and the properties of the end-surface as given from the starting surface. Creating the model depends highly on creating the initial polyhedron. This is what we want to discuss today.

There are several ways for constructing the initial polyhedron. One simple method is to use boolean algebra and some base methods from CAD/CAM. This would be building the model by working with simple building blocks and then considering unions and intersections, addition and substraction of sets. We will look at this method in more detail tomorrow. What this allows us to do is build the topology of the object in a constructive manner, and then let the subdivision process smooth the object.

This type of method is good for ab initio design problems, where the object is to be created from scratch with little or no constraints on the design. This is important to realize as the creation of the model is given by pure construction. This means that it is very hard to incorporate data into the method of construction. Building a model by union and intersection or addition and substraction of sets does not easily work by sampling points or data collection methods.

Today, we want to look at the following problem: given a collection of points to build a polyhedron with the given points forming the vertices. This type of problem is a called triangulization problem or more generally a mesh-generation problem in terms of a finite-element analysis problem. We will first consider the problem for points in a plane and then generalize it to points in space.

Typically, these types of problems involve subdividing the plane (or space) into tiles and then triangulating the tiles. Some of the terms involved and related to this problem are constructing a Dirichlet tesselation, a Voronoi diagram, or a Delaunay triangulation. This type of problem is more generally a base tool in computational geometry, constructing algorithms and methods to solve geometric problems.

38.1 Voronoi diagrams for Planar Sets of Points

Given a set of points $P = \{p_1, p_2, \cdots, p_n\}$ in a plane. A Voronoi diagram of P is the subdivision of the plane into n cells or tiles one for each point in P with the property that q belongs to the tile for point p_i if $\mathrm{dist}(p_i, q) < \mathrm{dist}(p_j, q)$ for each point $p_j \in P$ with $i \neq j$. The construction of the Voronoi diagram for P is given by for each point p_i as the intersection of planes $h(p_i, p_j)$ with $i \neq j$. In particular, let $V(p_i)$ be the tile for point p_i

and $h(p,q)$ be the open half plane containing p obtaining by bisecting the segment pq by the perpendicular bisector, see diagram below.

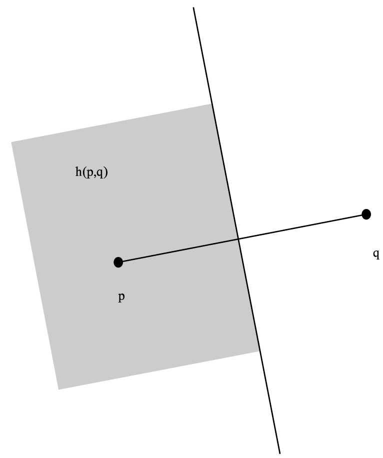

Figure 1: Half-plane $h(p,q)$

The tile $V(p_i)$ is defined as

$$V(p_i) = \cap_{1 \leq j \leq n,\ i \neq j}\ h(p_i, p_j)$$

see diagram below.

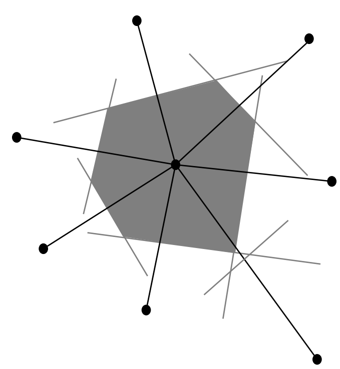

Figure 2: The tile $V(p_i)$ in a Voronoi diagram

This yields a complete Voronoi diagram (or Dirichlet tesselation) that looks like the diagram below.

Notice in this construction, the points p_i lie in the interior of each tile. This is not necessarily optimal for constructing a polyhedron. Instead, it would convenient if the points in the set P were the vertices of the polyhedron. This is accomplished by using the Voronoi diagram to construct a triangulization of the point set. First construct the Voronoi diagram, to get the tiles. Then we construct a triangulization by noting that generically only three bisectors meet at a point in the Voronoi diagram. We connect two points p_ip_j if the bisector p_ip_j is an edge of the tile $V(p_i)$ and $V(p_j)$. This forms a triangulization of the point set. This is called the Delaunay triangulization of the point set P, see diagram below.

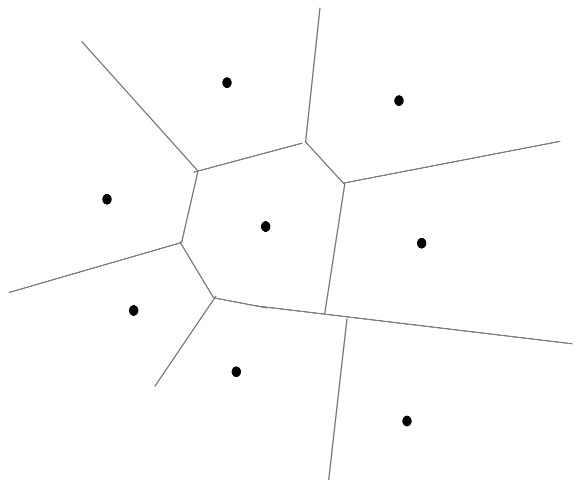

Figure 3: A Voronoi diagram

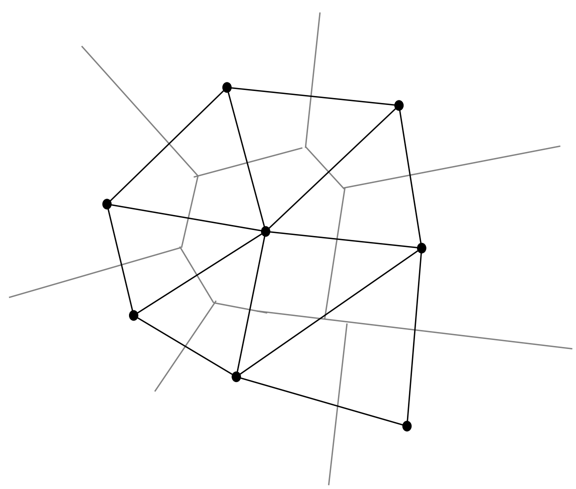

Figure 4: A Delaunay triangulation

There are exceptions to the above type of triangulization. We will not discuss these exceptions in great detail. This is important in computational geometry, but for our purposes can be avoiding by tweaking the data points.

38.2 Voronoi Diagrams in Space

To construct a triangulation for a collection of points $P = \{p_1, p_2, \cdots, p_n\}$ in space, we first note that the construction of a Voronoi diagram proceeds as before using half-spaces instead of half-planes. The construction of the Delanauy triangulation is more complicated. In principal, it is done the same connecting two points p_i and p_j if the tiles share a face. Except that they should not always be connected to form a polyhedral surface.

One method for constructing a Voronoi diagram is to project a subcollection of points onto a plane. Construct a Voronoi diagram for the projection to create a triangulation for the projection, and then lift the triangulation for the projection to the spatial points. The triangulations and then overlayed together as a collection of local triangulations and then fitted together to form a global triangulation, see diagrams below for constructing one.

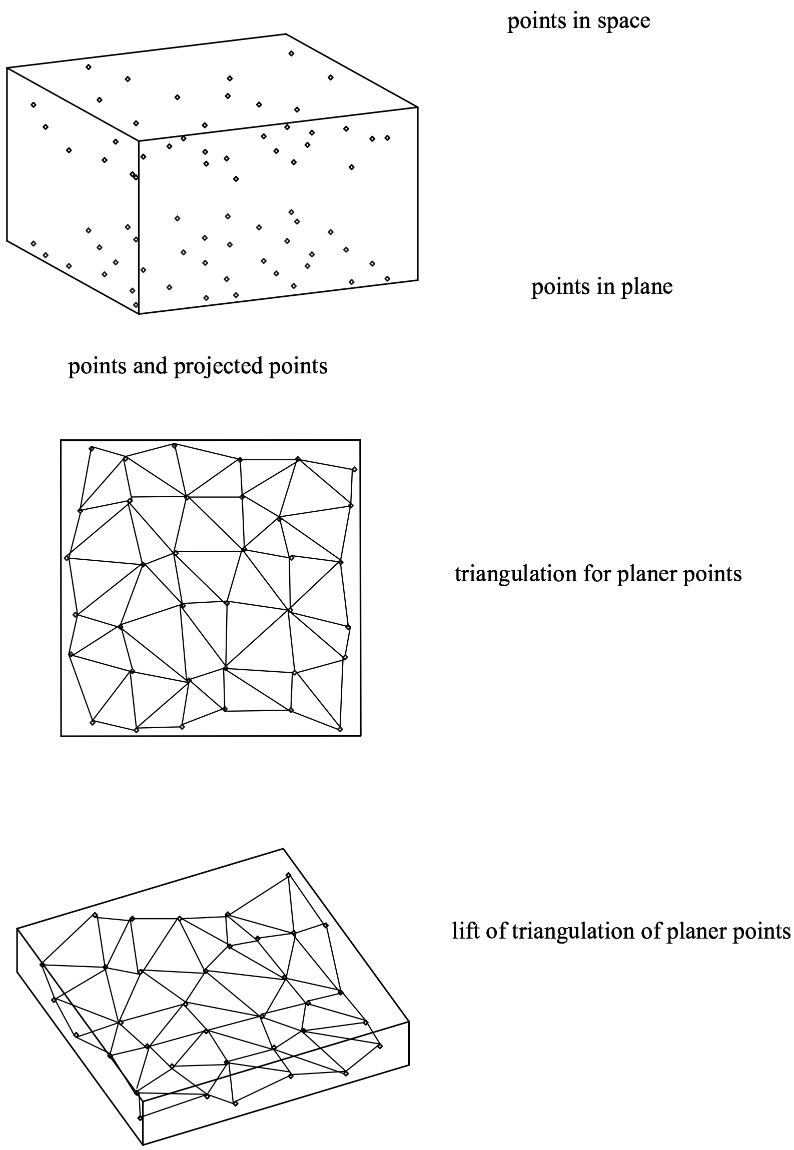

Figure 5: Constructing a triangulation for points on a surface

This works in general well for points on a surface, as points should be close to a plane at least when viewed in a small enough region. However it runs into some problems as the numbers may have to be large in order to effectively generate a mesh so the notion of closeness will easily work for projection. Luckily, most objects to be generated are sufficiently nice the topology of a sphere, and normally not too complicated.

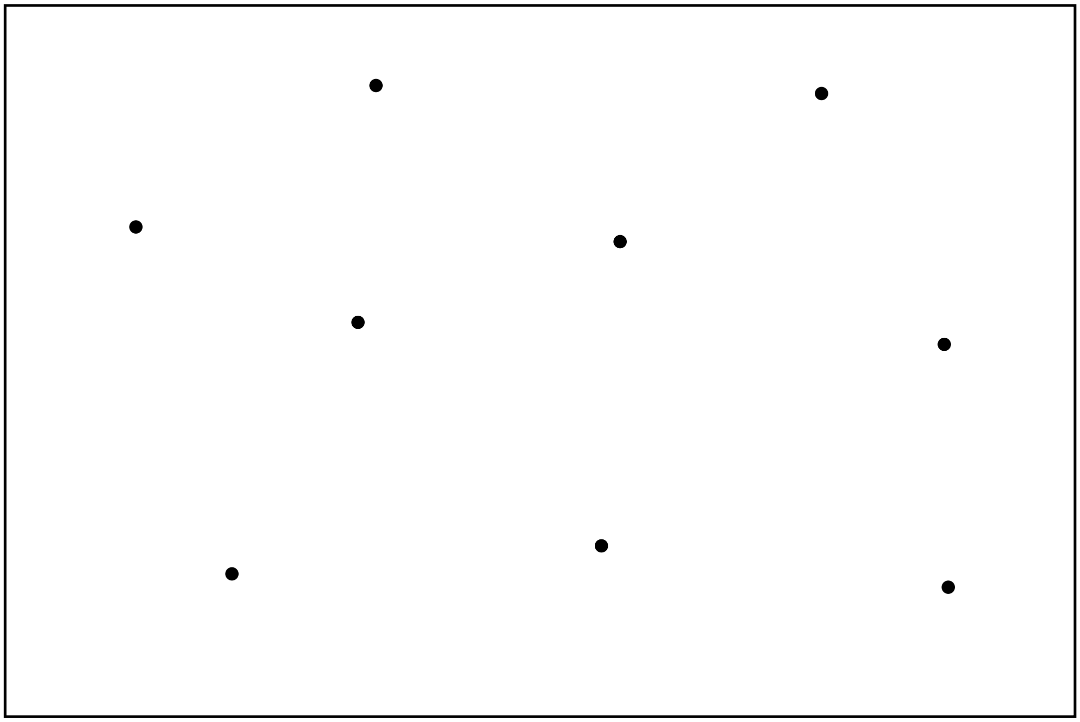

Figure 6: Construct Voronoi diagram and Delaunay triangulation

Exercies

1. Given the collection of points below, sketch the Voronoi diagram and the Delaunay triangulation. Ignoring points at infinity.