MA 323 Geometric Modelling
Course Notes: Day 36
Subdivision Surfaces

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Today, we continue our discussion of subdivision surfaces, by first looking in more detail at the midpoint method and the centroid method introduced in the last class. Specifically, we want to consider the size of the surface generated and the convergence of the methods. We also want to consider the smoothness of the surface generated. As of this point, we have only considered the base methods not any properties of the methods.

After considering the simply methods in some more detail, we introduce the Catmull-Clark method. This method is a generalization of bicubic B-Spline patches to a subdivision method. This method and the method we will introduce tomorrow, the Doo-Sabin Method, were introduced in 1978 in the Journal *Computer-Aided Design*. Both of these methods are generalizations of patch methods to arbitrary topology. Doo-Sabin surfaces are generalizations of biquadratic B-spline surfaces and Catmull-Clark surfaces are generalizations of bicubic B-spline surfaces. These methods are favored by some in the computer graphics industry. In particular, Pixar has used these types of surfaces to generate characters in their movies. Subdivision surfaces have the added benefit in computer graphics that it is easy to control the complexity of the object based on the size of the object in the picture. Fewer iterations are needed to render the object when it is distant. This also accounts for focus of objects that are not central to the scene.

35.1 General Setup

Let us start by setting some notation for the subdivision scheme in general. Let $\mathcal{P}$ represent the original polyhedron and $\mathcal{P}^i$ represent the polyhedron obtained after the ith iteration of the subdivision method. Our first goal for today is to look at the size of the polyhedron $\mathcal{P}^i$ in terms of the original polyhedron $\mathcal{P}$ and the iteration number i. Our second goal is to show or rather outline that there is a surface $\mathcal{S}$ that is the limiting surface of the polyhedrons $\mathcal{P}^i$, that is

$$\mathcal{S} = \lim_{i \to +\infty} \mathcal{P}^i.$$

The actual proof of the convergence requires substantial use of analysis as we need to prove the convergence of the limit. Finally, we want to look at whether the limiting surface is smooth in any sense of the word.

35.2 Midpoint Method

We first consider the size the polyhedra given by the midpoint method. Given that the original polyhedron $\mathcal{P}$ has size (F, E, V) meaning that the number of faces is F, the number of edges is E and the number of vertices is V. Letting the size of the ith iteration of $\mathcal{P}$

be given by (F_i, E_i, V_i), we find that after applying the midpoint method, that $V_{i+1} = E_i$, $F_{i+1} = F_i + V_i$, and $E_{i+1} = F_{i+1} + V_{i+1} - \xi = F_i + Vi + E_i - \xi = 2\,E_i$ as $\xi = F_i - E_i + V_i$ or $E_i = F_i + V_i - \xi$, where ξ is Euler characteristic of the surface $\xi = 2 - 2g$ where g is the genus of the surface or the number of holes in the surface. This gives

$$F_{i+1} = F + V + (2^{i-1} - 1)\,E$$
$$E_{i+1} = 2^i\,E$$
$$V_{i+1} = 2^{i-1}\,E$$

These relations may be seen easily by observing that the number of vertices of $\mathcal{P}^{i+1}$ is equal to the number of edges in $\mathcal{P}^\rangle$, and the number of faces in $\mathcal{P}^{i+1}$ is equal to the number of faces plus the number of vertices in $\mathcal{P}^i$. The number of edges in $\mathcal{P}^{i+1}$ is fixed by the topology of the surface, that is the Euler characteristic and Euler's formula for polyhedra.

To see that the midpoint method converges, we first note that the midpoint method satisfies the convex hull property. Meaning that the polyhedron $\mathcal{P}^{i+1}$ lies within the convex hull of the polyhedron $\mathcal{P}^i$. This means that that all the iterates $\mathcal{P}^i$ lie within the convex hull of $\mathcal{P}$ so that given that the original polyhedron is bounded (lies inside a sphere of large radius) all the iterates are also bounded. This means that the limiting process is confined to a given "compact set" and the iterates can not run off towards infinity. Formally this means that choosing a sequence of points $p_i \in V_i$ where V_i is the vertex set for $\mathcal{P}^i$ that the sequence of points has a convergent subsequence without using any facts about the choice of points itself and any facts about the method other than the convex hull property. To get the sequence to converge, we need to choose the sequence in a more specific manner. In particular, we choose p_i so that after iterate k they are generated from the same face from the polyhedron $\mathcal{P}^k$, meaning that p_k is incident to the face f' of $\mathcal{P}^k$ and p_{k+1} is incident to the face generated from f' (the face face in the midpoint midpoint), afterwards we continue to define p_i inductively in this manner. We claim that such a sequence converges. To see this, we need to show that the face generated by f' is smaller than the face f'. The look at a angle on the face (meaning three points that form two edges with a common vertex). The edge generated by this angle is half the distance between the nonincident vertices of the angle. This implies that the "diameter" of a face defined by as the maximum distance between two vertices incident to the face, satisfies the following property the diameter of the face generated by f' is strictly less than the diameter of the face f'. More importantly the length of the edges is less than $1/2$ the diameter of the face Therefore, the sequence constructed converges to a point on the surface $\mathcal{S}$. This shows basically that faces converge to points on the limiting surface. Since this means every sequence of sequence of iteratively defined faces converges, we can by continuity of each polyhedron conclude that the limiting surface exists.

Next we want to look at the smoothness of the surface. Let's suppose that the original polyhedron has planar faces. The iteration shows that the face faces are still then planar, and the limiting surface if smooth should have the the original plane as the tangent plane at the limiting point (the limit of the centroids of the iteratively defined face faces). To show that it is smooth, we need to show that the area of the faces do not decrease too fast. We note that the argument in the above paragraph shows that at best the area of the face decreases to $1/4$ the size of the original face and that is if the face was triangle. When the face has move than three sides, it decreases slower. This rate of convergence is slower enough to guarantee that the limiting point for the centroid is smooth. Showing nonplanar face face iterations converge to smooth points requires more work in particular one must show that the face face iterates get closer to planar.

35.3 Centroid Method

For the Centroid Method, we first note that the number of new faces is equal to the number of old faces plus the number of old edges plus the number of old vertices, the number of new edges is four times the number of old edges, and the number of new vertices is determined by the topology so that the Euler characteristic remains invariant. The number of faces is directly given by the method as it defines faces faces, edge faces, and vertex faces. The number of edges is also given directly from the method as each edge will generate two edge edges and two face edges for a total of four edges determined by each edge. In particular, each edge face is four sided and all edges are determined from this face. Therefore, we have

$$
\begin{aligned}
F_{i+1} &= F_i + E_i + V_i \\
E_{i+1} &= 4E_i \\
V_{i+1} &= \xi + E_{i+1} - F_{i+1} \\
&= F_i - E_i + V_i + 4E_i - F_i - E_i - V_i = 2E_i
\end{aligned}
$$

This gives

$$
\begin{aligned}
F_{i+1} &= F + V + \left(\tfrac{1}{2}\,4^i - 1\right)E \\
E_{i+1} &= 4^i\,E \\
V_{i+1} &= \tfrac{1}{2}\,4^i\,E
\end{aligned}
$$

Again we note that the Euler characteristic is important in determining the size of the iterates. It is important that the method preserves the Euler characteristic of the polyhedra. In particular that that method does not generate holes in the polyhedra.

The argument for the convergence of the limiting surface is very similar to the one for the midpoint method. Again the algorithm satisfies the convex hull property, so that the iteration can not run off towards infinity. Also the edge edges are guaranteed by construction to be less than the length of the original edges. The face edges are less than the distance between the centroids of neighboring faces. More importantly by the construction it is possible to estimate the rate of contraction of the distances to guarantee that the sequences of points generated on iterative face faces converge. More importantly all vertices by polyhedra structure belong to some face and thus every point on the limiting surface can be approximated by such a sequence.

The smoothness of the centroid method works in a similar manner to the midpoint method, with again the restriction to planar faces. The nonplanar faces also require the same subtlety in determine the convergence to a smooth point.

35.4 The Catmull-Clark Subdivision Method

In the Catmull-Clark method, rather than describing the generation globally, we construct the new polyhedron locally by describing what happens near each vertex v. In the procedure, we form new points (face points) f_j' corresponding to each face that v is adjacent to. We also form new points (edge points) e_j' corresponding to each edge for which v is an endpoint. Lastly, we form a new point vertex point v'.

- Each new face point f_j' is computed by finding the centroid of the face f_j.
- Each new edge point e_j' is the average of the endpoints of the edge and the face points f' that the edge is adjacent to.

- The new vertex point v' is computed as follows,

$$v' = \tfrac{1}{3}v + \tfrac{1}{3n} \sum_{j=1}^{n} e'_j + \tfrac{1}{3n} \sum_{j=1}^{n} f'_j,$$

where n is the number of edges (and faces) that surround the vertex point v. We are averaging the average of all faces adjacent to v, the average of all edges adjacent to v, and the vertex v.

We form new faces by considering loops (sequences of vertices) starting at the new vertex point v' to an edge point e'_1 to a face point f' to another edge point e'_2 back to the vertex point v'. The edge points e'_1, e'_2 must arise from edges e_1 and e_2 that share endpoint v. The face point f' should be the centroid of a face that shares the edges e_1 and e_2. Notice, that this algorithm produces faces that are four sided. Consider the diagram below, as an illustration.

Let's consider the growth in size of the polyhedron under this procedure, letting F be the number of original faces, E be the number of original edges, and V be the number of original vertices. Further, let the size of the ith iterate be (F_i, E_i, V_i). The algorithm implies that $V_{i+1} = F_i + E_i + V_i$ since each face, edge and vertex generates a new vertex. In addition, the face that each face is four sided implied that the number of edges is twice the number of current faces, i.e. $E_{i+1} = 2F_{i+1}$, since each edge belongs to two faces. The number of current faces is then given by the Euler characteristic $F_{i+1} - E_{i+1} + V_{i+1} = \xi = F_i - E_i + V_i$ or $-F_{i+1} + F_i + E_i + V_i = F_i - E_i + V_i$ which gives $F_{i+1} = 2E_i$. This described in summary below

$$F_{i+1} = 2E_i E_{i+1} \qquad = 2F_{i+1} = 4E_i$$
$$V_{i+1} = F_i + E_i + V_i$$

or

$$F_{i+1} = \tfrac{1}{2} 4^i E$$
$$E_{i+1} = 4^i E$$
$$V_{i+1} = F + V + (\tfrac{1}{2} 4^i - 1) E$$

which is very similar to the centroid method replacing the number of vertices with the number of faces.

Showing that the algorithm converges is a little more delicate. First, we note that like the midpoint method and the centroid method, the Catmull-Clark method satisfies the convex hull property, as each new vertex is arrived at via a convex combination of original vertices. This means that the limiting surface can not run off towards infinity. It takes a more work to show that the lengths of edges decrease, but once that is done then the convergence to a limiting surface proceeds as before.