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Optimal Smoothing Spline with Constraints on Its Derivatives

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***Abstract*— This paper considers the problem of designing optimal smoothing spline with constraints on its derivatives. The splines of degree k are constituted by employing normalized uniform B-splines as the basis functions. We then show that the l-th derivative of the spline can be obtained by using B-splines of degree $k-l$ with the control points computed as l-th difference of original control points. This yields systematic treatment of equality and inequality constraints over intervals on derivatives of arbitrary degree. Also, pointwise constraints can readily be incorporated. The problem of optimal smoothing splines with constraints reduce to convex quadratic programming problems. The effectiveness is demonstrated by numerical examples of approximations of probability distribution function and concave function, and trajectory planning with the constraints on velocity and acceleration.**

I. Introduction

The problem of constructing approximations to curves with constraints on the derivatives is important in a number of areas. A classical example is the problem of approximating the growth chart of an individual from birth to age 18. If classical smoothing splines are used there is no guarantee that the curve will be monotone. One looses a lot of credibility if the chart shows that the child has grown shorter at some period during its life. The problem of constructing monotone smoothing splines was solved in [4] using cubic smoothing splines but the construction was very specific to the cubic case. Meyer in [9] has also constructed monotone smoothing splines but her construction is also specific to the cubic case. In this paper we construct monotone smoothing splines of higher order using B-splines in conjunction with quadratic programming. This is important for it allows the one to take derivatives of the splines and to have good convergence properties of the derivatives. With cubic splines derivatives have quite degraded convergence properties. Splines with constraints on derivatives are also studied in [2], [13].

It is quite easy to construct splines with inequality constraints on the derivatives at points. This was done in [10], [14] and in book form in [5]. The problem reduces to a quadratic programming problem and is easily solved. The problem of imposing the constraints on an interval seems to lead to an infinite dimensional problem which seemed out of reach. In [4] the problem was solved as a dynamic programming problem but the method was special to the cubic case. Very recently Nagahara, [11], has given a suboptimal solution to the problem of constructing splines with derivative constraints using positive systems. In [11] the constraints are restricted to be of the form $x^{(j)}(t) \geq 0$. The solutions given in this paper include a solution to one basic problem of constructing splines with $x(t) \geq 0$ and $x''(t) \leq 0$. This is a problem frequently encountered in statistics.

This paper is organized as follows. In Section II, we briefly review B-splines and design methods of optimal smoothing splines. Then in Section III, we develop a systematic method for constructing optimal spline with constraints on its derivatives of arbitrary degree. We examine the performances of the proposed method by numerical examples in Section IV. Concluding remarks are given in Section V. All the Lemmas and Propositions are presented without proofs.

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II. Preliminaries

A polynomial spline $x(t)$ of degree k in an interval $\mathscr{D} = [t_0, t_m] \subset \mathbf{R}$ can be represented as

$$x(t) = \sum_{i=-k}^{m-1} \tau_i B_k(\alpha(t - t_i)), \tag{1}$$

by an appropriate choice of the weighting coefficients $\tau_i \in \mathbf{R}$ called control points [1]. Here, $B_k(t)$ is a normalized, uniform B-spline function of degree k, m is an integer, and $\alpha(>0)$ is a constant for scaling the interval between equally-spaced knot points t_i with

$$t_{i+1} - t_i = \frac{1}{\alpha}. \tag{2}$$

It is noted that employing a higher degree k of B-splines in (1) yields splines $x(t)$ of higher degree and thus allows us to design more complex curves. Also, for fixed k and the interval $[t_0, t_m]$, increasing the parameter α (i.e. smaller knot points spacing) gives us more flexibility of spline design since m (equivalently the number of control points) increases.

As preliminaries, we briefly review the basic problem of optimal splines based on normalized uniform B-splines.

A. Normalized Uniform B-Splines

Normalized uniform B-spline $B_k(t)$ of degree k is defined by

$$B_k(t) = \begin{cases} N_{k-j,k}(t-j) & j \leq t < j+1, \\ & j = 0, 1, \cdots, k \\ 0 & t < 0 \ \text{ or } \ t \geq k+1, \end{cases} \tag{3}$$

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and the basis elements $N_{j,k}(t)$ $(j=0,1,\cdots,k)$, $0 \le t \le 1$ are obtained recursively by the following algorithm:

Algorithm 1: Let $N_{0,0}(t) \equiv 1$ and, for $i=1,2,\cdots,k$, compute

$$\left\{\begin{array}{rcl} N_{0,i}(t) & = & \frac{1-t}{i} N_{0,i-1}(t) \\ N_{j,i}(t) & = & \frac{i-j+t}{i} N_{j-1,i-1}(t) + \frac{1+j-t}{i} N_{j,i-1}(t), \\ & & \qquad\qquad j=1,\cdots,i-1 \\ N_{i,i}(t) & = & \frac{t}{i} N_{i-1,i-1}(t). \end{array}\right. \quad (4)$$

Thus, $B_k(t)$ is a piece-wise polynomial of degree k with integer knot points and is $k-1$ times continuously differentiable.

For the sake of later reference, we introduce $(k+1)$-dimensional vectors $N_k(t)$ and $h_k(t)$ as

$$\begin{array}{rcl} N_k(t) & = & \left[\begin{array}{cccc} N_{0,k}(t) & N_{1,k}(t) & \cdots & N_{k,k}(t) \end{array}\right]^T \quad (5) \\ h_k(t) & = & \left[\begin{array}{cccc} t^k & t^{k-1} & \cdots & 1 \end{array}\right]^T. \quad (6) \end{array}$$

Then $N_k(t)$ is written as

$$N_k(t) = S_k h_k(t), \quad (7)$$

where $S_k \in \mathbf{R}^{(k+1)\times(k+1)}$ is a matrix whose i-th row consists of the coefficients of polynomial $N_{i-1,k}(t)$. It can be shown that the matrix S_k can be obtained by the following recursive algorithm. Letting $S_0 = 1$, compute $S_i \in \mathbf{R}^{(i+1)\times(i+1)}$ for $i = 1,2,\cdots,k$ by

$$S_i = \frac{1}{i}\left(\left[\begin{array}{cc} 0_{i+1} & \Gamma_i S_{i-1} \end{array}\right] + \left[\begin{array}{cc} \Delta_i S_{i-1} & 0_{i+1} \end{array}\right]\right), \quad (8)$$

where the matrices $\Gamma_i, \Delta_i \in \mathbf{R}^{(i+1)\times i}$ are defined as

$$\Gamma_i = \left[\begin{array}{ccccc} 1 & & & & \\ i-1 & 2 & & & \\ & i-2 & 3 & & \\ & & \ddots & \ddots & \\ & & & 1 & i \\ & & & 0 & 0 \end{array}\right], \Delta_i = \left[\begin{array}{ccccc} -1 & & & & \\ 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \\ & & & & 1 \end{array}\right]. \quad (9)$$

Here the empty spaces denote zero entries.

B. Optimal Smoothing Splines

The control points τ_i in (1) are typically determined by the theory of smoothing splines (see, e.g. [7] for details). Suppose that we are given a set of data

$$\{(s_i, d_i) \ : \ s_i \in [t_0, t_m],\ d_i \in \mathbf{R},\ i=1,2,\cdots,N\}, \quad (10)$$

and let $\tau \in \mathbf{R}^M$ $(M = m+k)$ be the weight vector defined by

$$\tau = \left[\begin{array}{cccc} \tau_{-k} & \tau_{-k+1} & \cdots & \tau_{m-1} \end{array}\right]^T. \quad (11)$$

Then a standard problem is to find such a τ minimizing the cost function

$$J(\tau) = \lambda \int_{t_0}^{t_m} \left(x^{(2)}(t)\right)^2 dt + \sum_{i=1}^{N} w_i \left(x(s_i) - d_i\right)^2, \quad (12)$$

where $\lambda(>0)$ is a smoothing parameter, and $w_i (0 \le w_i \le 1)$ are weights for approximation errors.

Introducing a vector $b(t) \in \mathbf{R}^M$,

$$\begin{array}{rcl} b(t) & = & \left[\begin{array}{cc} B_k(\alpha(t-t_{-k})) & B_k(\alpha(t-t_{-k+1})) \end{array}\right. \\ & & \qquad \left.\begin{array}{cc} \cdots & B_k(\alpha(t-t_{m-1})) \end{array}\right]^T, \quad (13) \end{array}$$

the spline $x(t)$ in (1) is written as $x(t) = \tau^T b(t)$. Then, the cost $J(\tau)$ in (12) is rewritten as a quadratic function of τ,

$$J(\tau) = \tau^T G \tau - 2 g^T \tau + r, \quad (14)$$

with

$$G = \lambda Q + BWB^T, \quad g = BWd, \quad r = d^T W d. \quad (15)$$

Here $Q \in \mathbf{R}^{M\times M}$ is a Gramian defined by

$$Q = \int_{t_0}^{t_m} \frac{d^2 b(t)}{dt^2} \frac{d^2 b^T(t)}{dt^2} dt. \quad (16)$$

The matrices $B \in \mathbf{R}^{M\times N}$, $W \in \mathbf{R}^{N\times N}$, $d \in \mathbf{R}^N$ are defined by

$$\begin{array}{rcl} B & = & \left[\begin{array}{cccc} b(s_1) & b(s_2) & \cdots & b(s_N) \end{array}\right], \quad (17) \\ W & = & \mathrm{diag}\{w_1,\ w_2.\ \cdots,\ w_N\}, \quad (18) \\ d & = & \left[\begin{array}{cccc} d_1 & d_2 & \cdots & d_N \end{array}\right]^T. \quad (19) \end{array}$$

Notice here that G in (15) is positive-semidefinite, i.e. $G \ge 0$, since $\lambda > 0$, $Q \ge 0$ and $W \ge 0$, and hence $J(\tau)$ is a convex function. Thus, if there are no constraints, the optimal solution is given as a solution of linear algebraic equations,

$$G\tau = g. \quad (20)$$

Note that (20) has at least one solution and obviously the solution is unique if and only if $G > 0$. In addition, once the parameters k and m (or $M(=m+k)$) are fixed, the size of the algebraic equation (20) remains the same regardless of the number of data N. The Gramian $Q \in \mathbf{R}^{M\times M}$ in (16) can be computed explicitly by using B-splines (see e.g. [7]).

On the other hand, a given function $f(t)$, $t \in [t_0, t_m]$ can also be approximated by smoothing splines, in which case the following cost function is used.

$$J(\tau) = \lambda \int_{t_0}^{t_m} \left(x^{(2)}(t)\right)^2 dt + \int_{t_0}^{t_m} \left(x(t) - f(t)\right)^2 dt. \quad (21)$$

Similarly as above, this cost function is rewritten as

$$J(\tau) = \tau^T G \tau - 2 g^T \tau + f_c, \quad (22)$$

where G, g and f_c denote

$$G = \lambda Q + Q_0, \quad g = \int_{t_0}^{t_m} b(t) f(t) dt, \quad f_c = \int_{t_0}^{t_m} f^2(t) dt. \quad (23)$$

Here, $Q_0 \in \mathbf{R}^{M\times M}$ is defined as

$$Q_0 = \int_{t_0}^{t_m} b(t) b^T(t) dt. \quad (24)$$

Obviously, it holds that $G > 0$ in (23) since $Q_0 > 0$ (see [6]), hence $J(\tau)$ in (22) is strictly convex in τ and unique optimal solution exists.

III. Spline with Constraints on Derivatives

For the optimal smoothing spline $x(t)$ of degree k as described in the previous section, we impose the following condition on its l-th derivative

$$x^{(l)}(t) \ge c \quad \forall t \in [t_j, t_{j+1}], \quad (25)$$

where $0 \le l \le k$ and c is a given constant. Setting $c = 0$ yields monotone splines.

Moreover, we generalize the constant c in (25) to some function of t, e.g. a linear function as

$$x^{(l)}(t) \geq pt + q \quad \forall t \in [t_j, t_{j+1}]. \tag{26}$$

Note that such constraints for each knot point interval $[t_j, t_{j+1}]$ allow us more flexible treatment of constraints over intervals, such as the curve being convex on some interval and concave on another. Also, the above inequality '$\geq$' can be readily replaced with '$\leq$' or equality '$=$', as we will see in the subsequent development.

Our problem here is to express such constraints in terms of the control points τ_i. In the sequel, we first develop basic formula for derivatives of splines.

A. Formula for Derivatives of Spline

Since $x(t)$ is a piece-wise polynomial of degree k, we examine the polynomial in each interval $[t_j, t_{j+1}]$ for $j = 0, 1, \cdots, m-1$. Focusing on the interval $[t_j, t_{j+1}]$, the spline $x(t)$ in (1) is written as

$$x(t) = \sum_{i=j-k}^{j} \tau_i B_k(\alpha(t - t_i)). \tag{27}$$

Using (3), we then get

$$x(t) = \sum_{i=0}^{k} \tau_{j-k+i} N_{i,k}(\alpha(t - t_j)), \quad t \in [t_j, t_{j+1}], \tag{28}$$

and it depends on only the $k+1$ weights τ_{j-k}, τ_{j-k+1}, $\cdots$, τ_j. Moreover, by introducing a new variable u,

$$u = \alpha(t - t_j), \tag{29}$$

the interval $[t_j, t_{j+1}]$ in t is normalized to $[0, 1]$ in u , and we may write $x(t)$ as $\hat{x}(u)$,

$$\hat{x}(u) = \sum_{i=0}^{k} \tau_{j-k+i} N_{i,k}(u), \quad u \in [0, 1]. \tag{30}$$

Letting $\tau_{[j-k,j]} \in \mathbf{R}^{k+1}$ be a vector

$$\tau_{[j-k,j]} = \begin{bmatrix} \tau_{j-k} & \tau_{j-k+1} & \cdots & \tau_j \end{bmatrix}^T, \tag{31}$$

and using (7), we rewrite $\hat{x}(u)$ in (30) as

$$\hat{x}(u) = \tau_{[j-k,j]}^T N_k(u). \tag{32}$$

In general, the l-th derivative $x^{(l)}(t)$ for $t \in [t_j, t_{j+1}]$ is expressed in terms of $u \in [0, 1]$ in (29) by

$$x^{(l)}(t) = \alpha^l \hat{x}^{(l)}(u), \tag{33}$$

with

$$\hat{x}^{(l)}(u) = \sum_{i=0}^{k} \tau_{j-k+i} N_{i,k}^{(l)}(u) = \tau_{[j-k,j]}^T N_k^{(l)}(u). \tag{34}$$

Now we prepare a lemma where derivatives of basis elements $N_k(t)$ of splines in (5) are related to lower order elements by the matrix Δ_i in (9). Here we define a matrix $\Delta_{[i_1,i_2]} \in \mathbf{R}^{(i_1+1)\times i_2}$ for $i_1 \geq i_2$ by

$$\Delta_{[i_1,i_2]} = \prod_{\nu=i_1}^{i_2} \Delta_\nu = \Delta_{i_1} \Delta_{i_1-1} \cdots \Delta_{i_2}. \tag{35}$$

Lemma 1: The first derivative of vector $N_i(t)$ is given by

$$N_i^{(1)}(t) = \Delta_i N_{i-1}(t), \quad i = 1, 2, \cdots \tag{36}$$

and hence the l-th derivative by

$$N_i^{(l)}(t) = \Delta_{[i,i-l+1]} N_{i-l}(t). \tag{37}$$

This lemma shows that the l-th derivative $\hat{x}^{(l)}(u)$ in (34) can be represented by the basis elements $N_{i,k'}(u)$, $(i = 0, 1, \cdots, k')$ of degree k', where

$$k' = k - l. \tag{38}$$

Thus we let

$$\hat{x}^{(l)}(u) = \sum_{i=0}^{k'} \phi_{j-k'+i} N_{i,k'}(u) = \phi_{[j-k',j]}^T N_{k'}(u), \tag{39}$$

where $\phi_{[j-k',j]} \in \mathbf{R}^{k'+1}$ is defined by

$$\phi_{[j-k',j]} = \begin{bmatrix} \phi_{j-k'} & \phi_{j-k'+1} & \cdots & \phi_j \end{bmatrix}^T. \tag{40}$$

From (34), (37) and (39), the vector $\phi_{[j-k',j]}$ is related to $\tau_{[j-k,j]}$ by

$$\phi_{[j-k',j]} = \Delta_{[k,k-l+1]}^T \tau_{[j-k,j]}. \tag{41}$$

The coefficients ϕ_i in (39) are determined in terms of the control points τ_i in (30) as follows, where we introduced a shift operator z such that

$$z^{i'} \tau_i = \tau_{i+i'}. \tag{42}$$

Lemma 2: The l-th derivative $x^{(l)}(t)$ of spline $x(t)$ in (27) is expressed as spline in (33) and (39), where the control points ϕ_i in (39) are given by

$$\phi_i = A_{j-i,l}(z) \tau_j, \quad i = j-k', j-k'+1, \cdots, j, \tag{43}$$

with $A_{n,l}(z) = z^{-n}(1 - z^{-1})^l$.

Remark 1: Noting that $1 - z^{-1}$ is a difference operator as $(1 - z^{-1})\tau_j = \tau_j - \tau_{j-1}$, the term $(1 - z^{-1})^l \tau_j$ in $A_{j-i,l}(z)\tau_j$ gives the l-th backward difference beginning with τ_j. Then the remaining factor $z^{-(j-i)}$ for $i = j-k', j-k'+1, \cdots, j$ gives all the l-th difference formed from the sequence of $k+1$ control points $\{\tau_{j-k},\ \tau_{j-k+1},\ \cdots\ \tau_j\}$.

Using (33), (39) and $u = \alpha(t - t_j)$ in (29), we get

$$x^{(l)}(t) = \alpha^l \sum_{i=0}^{k'} \phi_{j-k'+i} N_{i,k'}(\alpha(t - t_j)), \quad t \in [t_j, t_{j+1}]. \tag{44}$$

The l-th derivative of $x(t)$ in (1) is then expressed in terms of B-splines as

$$x^{(l)}(t) = \alpha^l \sum_{i=-k'}^{m-1} \phi_i B_{k'}(\alpha(t - t_i)), \quad t \in [t_0, t_m]. \tag{45}$$

By Lemma 2, we thus have a nice property that the l-th derivative $x^{(l)}(t)$ of spline $x(t)$ is determined by the l-th difference ϕ_i of control points τ_i for $x(t)$.

B. Constraints on Derivatives

Now we are in a position to state the constraint in (25) in terms of the control points τ_i.

Proposition 1: If the control points ϕ_i given by (43) satisfy

$$\phi_i \geq \frac{c}{\alpha^l}, \quad i = j-k', j-k'+1, \cdots, j, \tag{46}$$

then the spline $x(t)$ satisfies the constraint (25).

Introducing a vector $\mathbf{1}_i = [1\ 1\ \cdots\ 1]^T \in \mathbf{R}^i$, the constraint (46) is written as

$$\phi_{[j-k',j]} \geq \frac{c}{\alpha^l}\mathbf{1}_{k'+1}, \tag{47}$$

and (41) gives the expression in terms of original control points τ_i as

$$\Delta^T_{[k,k+1-l]}\tau_{[j-k,j]} \geq \frac{c}{\alpha^l}\mathbf{1}_{k'+1}. \tag{48}$$

This constraint is easily extended to knot point interval of arbitrary length, say $[t_j, t_{j+n}]$ for $n \geq 1$, as

$$\Delta^T_{[k+n-1,k+n-l]}\tau_{[j-k,j+n-1]} \geq \frac{c}{\alpha^l}\mathbf{1}_{k'+n}. \tag{49}$$

Example 1: If we impose the constraint (25) over the entire interval, namely $x^{(l)}(t) \geq c,\ \forall t \in [t_0, t_m]$, then letting $j = 0$ and $n = m$ in (49) yields the constraint on the control point vector τ as

$$\Delta^T_{[k+m-1,k+m-l]}\tau \geq \frac{c}{\alpha^l}\mathbf{1}_{k'+m} \tag{50}$$

since $\tau_{[-k,m-1]} = \tau$ by (11).

Example 2: Since the first and second derivatives of $x(t)$ are of particular interest, we introduce simpler notations. Letting $D^1_i = \Delta^T_{[i,i]} = \Delta^T_i$ and $D^2_i = \Delta^T_{[i,i-1]} = (\Delta_i\Delta_{i-1})^T$, we have

$$D^1_i = \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix} \quad (\in \mathbf{R}^{i\times(i+1)}) \tag{51}$$

and

$$D^2_i = \begin{bmatrix} 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \end{bmatrix} \quad (\in \mathbf{R}^{i\times(i+2)}) \tag{52}$$

where the empty spaces denote zero entries.

Using these matrices, the condition (50) for $l = 1$ and $l = 2$ are rewritten as $D^1_{k+m-1}\tau \geq \frac{c}{\alpha}\mathbf{1}_{k+m-1}$ and $D^2_{k+m-2}\tau \geq \frac{c}{\alpha^2}\mathbf{1}_{k+m-2}$, respectively.

Next we generalize the constraint in (25) from constant c to some function of t. Specifically, we consider the constraint

$$x^{(l)}(t) \geq v(t) \quad \forall t \in [t_j, t_{j+1}] \tag{53}$$

where we assume that $v(t)$ is itself a spline of degree k' and expressed in terms of B-splines as

$$v(t) = \sum_{i=-k'}^{m-1} \mu_i B_{k'}(\alpha(t-t_i)). \tag{54}$$

Then $v(t)$ in the interval $[t_j, t_{j+1}]$ is written as

$$v(t) = \sum_{i=j-k'}^{j} \mu_i B_{k'}(\alpha(t-t_i)) \quad t \in [t_j, t_{j+1}] \tag{55}$$

and, similarly as (27)-(30) with $k = k'$, we get $v(t) = \hat{v}(u)$ with $u = \alpha(t - t_j)$ and

$$\hat{v}(u) = \sum_{i=0}^{k'} \mu_{j-k'+i} N_{i,k'}(u), \quad u \in [0,1]. \tag{56}$$

Proposition 1 is now generalized as follows.

Proposition 2: If the control points ϕ_i given by (43) satisfy

$$\phi_i \geq \frac{\mu_i}{\alpha^l}, \quad i = j-k', j-k'+1, \cdots, j, \tag{57}$$

then (53) holds.

Example 3: A simple but useful example of $v(t)$ in (54) is a linear function in t, say,

$$v(t) = p(t-t_0) + q. \tag{58}$$

This function is realized by the control points μ_i given by

$$\mu_i = \frac{1}{2\alpha}\left(2i + k' + 1\right)p + q, \quad i = j-k', j-k'+1, \cdots, j. \tag{59}$$

For example, by setting $l = 0$ in Proposition 2, the condition $\phi_i \geq \mu_i, \quad i = j-k, j-k+1, \cdots, j$ with the above μ_i guarantees that the function $x(t)$ satisfies

$$x(t) \geq p(t-t_0) + q \quad \forall t \in [t_j, t_{j+1}]. \tag{60}$$

Also, for the inequality to hold on the entire interval $[t_0, t_m)$, we simply let $j = 0, 1, \cdots, m-1$ yielding $\phi_i \geq \mu_i$ for $i = -k, -k+1, \cdots, m-1$.

As we have seen, we now have a method of describing equality and inequality constraints on all the derivatives of splines over basic knot point interval, and hence any knot point intervals. Moreover, we can describe various types of constraints at isolated points and integral values of splines [8]. All these constraints are expressed as linear constraints on the control points.

Thus, we can now design optimal smoothing splines by minimizing the convex quadratic cost $J(\tau)$ as shown in (14) and (22), whereas a number of constraints on the splines are expressed as linear constraints on τ, either equality or inequality or both. A general form of problems is then

$$\min_{\tau \in \mathbf{R}^M} J(\tau) = \frac{1}{2}\tau^T G \tau + g^T \tau \tag{61}$$

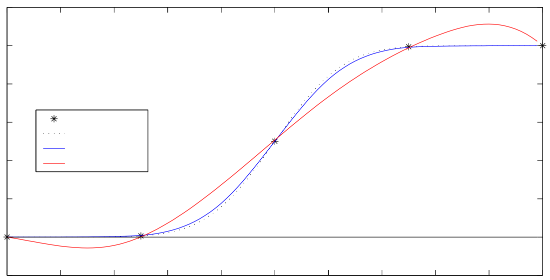
subject to the constraints of the form

$$A\tau = d,\ f_1 \leq E\tau \leq f_2,\ h_1 \leq \tau \leq h_2, \tag{62}$$

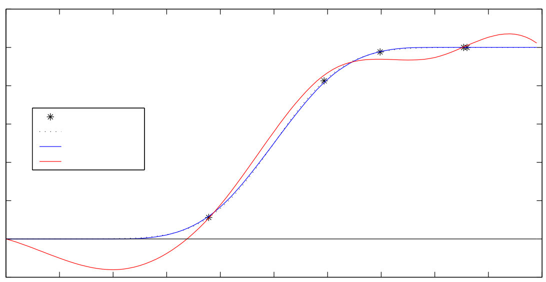
for some matrices and vectors of appropriate dimensions. A very efficient numerical algorithm is available for this purpose (see, e.g. [12]).

IV. Numerical Examples

We examine the design method presented in the previous sections numerically. As examples, we consider the problems of approximating probability distribution function and nonnegative concave function, and trajectory planning. Either cubic (i.e. $k = 3$) or quintic ($k = 5$) splines are used.



(a) Equally spaced data points



(b) Randomly spaced data points

Fig. 1. Approximation of Gaussian probability distribution function $f(t)$ from its sampled data * by cubic smoothing splines with and without the monotonicity constraints (64), denoted $x(t)$ and $x_0(t)$ respectively.

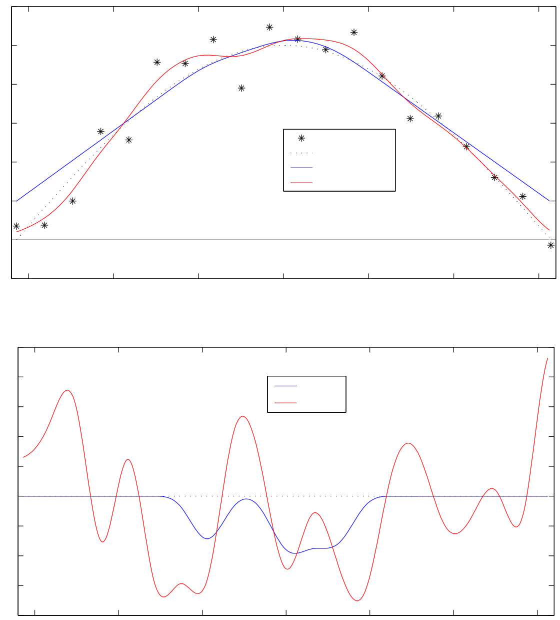


Fig. 2. Approximation of nonnegative concave function $f(t)$ from its noisy samples * by quintic smoothing splines with and without the constraints on the second derivative in (66), denoted $x(t)$ and $x_0(t)$ respectively. The splines $x(t)$ and $x_0(t)$ (top) and their second derivatives (bottom).

A. Approximation of Probability Distribution Functions

Let $f(t)$ be the Gaussian probability distribution function with zero mean and unit variance. We then approximate $f(t)$ in the interval $[t_0, t_m] = [-5, +5]$ from its samples $d_i = f(s_i),\ i = 1, 2, \cdots, N$. For $N = 5$, we consider the two cases where the sampling points s_i are equally spaced and randomly spaced. It is noted that the knot points are equally spaced and the same in both cases.

With $k = 3$, $\alpha = 1/2$ and $m = 5$ in (1), we reconstruct $f(t)$ as an optimal smoothing spline $x(t)$ based on the criterion (12) with $\lambda = 10^{-3}$ and $w_i = 1/N$. We impose the equality constraints at the boundaries,

$$x(-5) = 0,\ x(5) = 1, \tag{63}$$

and inequality constraints on the first derivative as

$$x^{(1)}(t) \geq 0\ \forall t \in [-5, 5]. \tag{64}$$

For specifying the constraints in terms of the control point vector τ, we use the method in [8] for (63) and the method developed in Section III for (64).

The results are shown in Figure 1, where the data points (s_i, d_i) are shown by asterisks *, and $f(t)$ and the designed spline $x(t)$ are plotted in black dotted line and blue solid line respectively. Also we showed in red solid line an optimal smoothing spline $x_0(t)$ obtained without the constraints (64). We see that the curve $x(t)$ closely approximates $f(t)$ while maintaining the monotone nondecreasing property specified as (64), which is not the case with the curve $x_0(t)$.

B. Approximation of Concave Function

We approximate the following concave function

$$f(t) = \cos(t) \tag{65}$$

in $[t_0, t_m] = [-\pi/2, \pi/2]$ by optimal smoothing spline $x(t)$ with the constraints

$$x(t) \geq 0,\quad x^{(2)}(t) \leq 0 \quad \forall t \in [-\pi/2, \pi/2]. \tag{66}$$

The data (s_i, d_i) are generated by sampling $f(t)$ at $20 (= N)$ equally spaced points s_i in $[-\pi/2, \pi/2]$ with additive Gaussian white noise of zero mean and standard deviation 0.1.

The design parameters for smoothing are set as $k = 5$, $m = 20$, $\lambda = 0.0001$ and $w_i = 1/N$. The constraint $x(t) \geq 0\ \forall t$ in (66) is realized by $\tau \geq 0$ (see [8]), whereas $x^{(2)}(t) \leq 0\ \forall t$ is realized by $D^2_{M-2}\tau \leq 0$ for the matrix D^2_i defined in (52).

The results are shown in Figure 2, where $x_0(t)$ is an optimal smoothing spline obtained without the constraint $x^{(2)}(t) \leq 0\ \forall t$. We observe that the desired results are obtained by including the constraints on second derivative.

C. Trajectory Planning

We consider a trajectory planning problem with equality and inequality constraints [3]. The time interval of interest is $[t_0, t_m] = [0, 1]$, and the initial and terminal conditions are set as

$$x(0) = x^{(1)}(0) = x^{(2)}(0) = 0,\ \ x(1) = 1,\ x^{(1)}(1) = x^{(2)}(1) = 0. \tag{67}$$

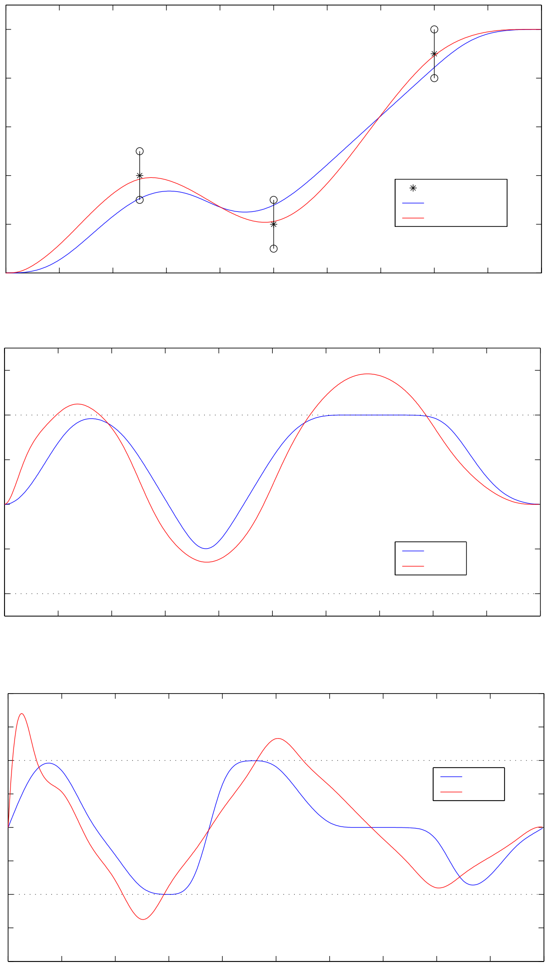


Fig. 3. Planned trajectories $x(t)$ (top), $x^{(1)}(t)$ (middle) and $x^{(2)}(t)$ (bottom) by quintic smoothing splines, and their counterparts $x_0(t)$, $x_0^{(1)}(t)$ and $x_0^{(2)}(t)$ without the constraints in (69).

We require the trajectory $x(t)$ to pass through the intervals $[\underline{x}_i, \bar{x}_i]$ at the three time instants $s_1 = 0.25,\ s_2 = 0.5,\ s_3 = 0.8$, namely

$$\underline{x}_i \le x(s_i) \le \bar{x}_i,\ i = 1,2,3, \tag{68}$$

with $[\underline{x}_1, \bar{x}_1] = [0.3, 0.5]$, $[\underline{x}_2, \bar{x}_2] = [0.1, 0.3]$, and $[\underline{x}_3, \bar{x}_3] = [0.8, 1.0]$. Moreover, the magnitudes of the velocity and acceleration are limited for the entire interval of time as

$$|x^{(1)}(t)| \le 2, \quad |x^{(2)}(t)| \le 20,\ \forall t \in [0,1]. \tag{69}$$

For designing the smoothing spline by the criterion in (12), we use the mid points of each interval in (68) as the data points, namely $d_i = (\underline{x}_i + \bar{x}_i)/2$ for $i = 1,2,3\ (= N)$, and thus $(s_1, d_1) = (0.25, 0.4)$, $(s_2, d_2) = (0.5, 0.2)$, and $(s_3, d_3) = (0.8, 0.9)$ in (10). The design parameters are $k = 5$, $\alpha = 20$ and $m = 20$ in (1), and $\lambda = 10^{-5}$ and $w_i = 1/N = 1/3$ in (12).

Figure 3 shows the planned trajectory $x(t)$ and its derivatives $x^{(1)}(t)$ and $x^{(2)}(t)$ in blue lines. The red lines show the optimal splines $x_0(t)$ and its derivatives obtained without the velocity and acceleration constraints (69). We see that the trajectory $x(t)$ satisfies all the constraints.

V. Concluding Remarks

We presented a systematic method for designing optimal smoothing splines with equality and/or inequality constraints on their derivatives over intervals. The splines of degree k are constituted employing normalized uniform B-splines as the basis functions, and hence the central issue is to determine an optimal vector τ of the so-called control points. The l-th derivative of the spline are obtained by using B-splines of degree $k - l$ with the control points computed as l-th difference of original control points in τ. This yielded systematic treatments and solutions for problems with equality and inequality constraints over intervals on derivatives of arbitrary degree. Also, pointwise constraints can readily be incorporated. We demonstrated the effectiveness of the design method by numerical examples, namely, approximations of Gaussian distribution function and concave function, and trajectory planning with the constraints on velocity and acceleration.

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