

# MA 323 Geometric Modelling

## Course Notes: Day 14

### Properties of Bezier Curves

David L. Finn

In this section, we discuss the geometric properties of Bezier curves. These properties are either implied directly from de Casteljau's algorithm or properties of the Bernstein polynomials. We have indirectly mentioned a few of these properties in class, with a few of the properties obvious from the algorithm and Bezier form. However, other properties are far from obvious and require some deep thought.

We list the more obvious properties and provide the direct reasons for these properties, with little discussion. The more deep properties

#### 14.1 Endpoint Interpolation

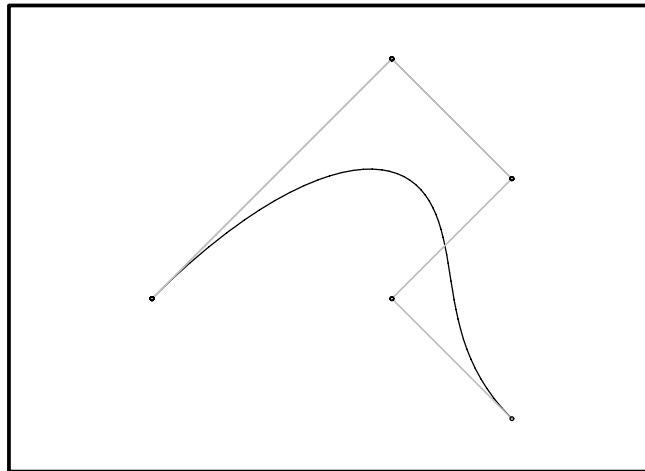


Figure 1: Bezier curve with endpoint interpolation

A Bezier curve always passes through the first and last control points. This is a property of de Casteljau's algorithm. When  $t = 0$ , from the definition of de Casteljau's algorithm  $p_i^{j+1} = p_i^j$  so  $p_0^n = p_0^{n-1} = \dots = p_0^1 = p_0^0 = p_0$ . In addition, when  $t = 1$ , we have  $p_i^{j+1} = p_{i+1}^j$  so  $p_0^n = p_1^{n-1} = \dots = p_{n-1}^1 = p_n^0 = p_n$ . This is also a direct consequence of the basis function approach by observing the following facts about the Bernstein polynomials  $0 \leq \beta_i^n(t) \leq 1$  for  $0 \leq t \leq 1$ ,  $0 < \beta_i^n(t) < 1$  for  $0 < t < 1$ ,  $\sum_{i=0}^n \beta_i^n(t) = 1$  and  $\beta_0^n(0) = 1 = \beta_n^n(1)$ . These facts imply that  $c(0) = p_0$  and  $c(1) = p_n$ . It is possible to arrange a Bezier curve to pass through other control points, but generically this will not happen.

## 14.2 Prescribed Tangent Lines at the Endpoints:

A Bezier curve has prescribed tangent lines at the control points  $p_0$  and  $p_n$ . The tangent lines being given by the line segments  $p_0p_1$  and  $p_{n-1}p_n$ . To see this, we note that  $c(t) = p_0^n(t) = (1-t)p_0^{n-1}(t) + tp_1^{n-1}(t)$  then differentiating we have  $c'(t) = p_1^{n-1}(t) - p_0^{n-1}(t) + (1-t)(p_0^{n-1})'(t) + t(p_1^{n-1})'(t)$ . Evaluate when  $t = 0$ , we have  $c'(0) = p_1^{n-1}(0) - p_0^{n-1}(0) + (p_0^{n-1})'(0)$ . From the endpoint interpolation,  $p_0^{n-1}(0) = p_0$  and  $p_1^{n-1}(0) = p_1$ . Induction then leads to  $(p_0^{n-k})'(t) = p_1^{n-k-1}(t) - p_0^{n-k-1}(t) + (p_0^{n-k-1})'(t)$  and  $(p_0^{n-k})'(0) = p_1 - p_0 + (p_0^{n-k-1})'(0)$ . Eventually, we are reduced to the linear case  $(p_0^1)'(t) = p_1^0 - p_0^0$ . Then back-substituting yields  $(p_0^n)'(0) = n(p_1 - p_0)$ . A similar argument at the other endpoint yields  $p_0^n(1) = n(p_n - p_{n-1})$ . Alternatively, one use the basis function approach and note from the derivative properties of a Bezier curve that

$$c'(t) = n \sum_{i=0}^{n-1} \beta_i^{n-1}(t) \Delta_i p.$$

The endpoint interpolation property above implies that  $c'(0) = n\Delta_0 p = n(p_1 - p_0)$  and  $c'(1) = n\Delta_{n-1} p = n(p_n - p_{n-1})$ .

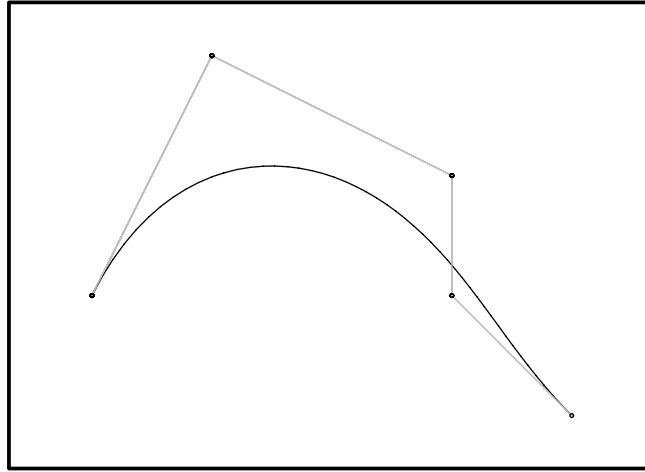


Figure 2: Prescribed Tangent Lines at the Endpoints

## 14.3 Symmetry:

Reversing the order of the control points produces the same curve. This is a direct consequence of the change of variables  $s = 1 - t$  that converts the straight line  $(1-t)A + tB$  which parameterizes the line from  $A$  when  $t = 0$  to  $B$  when  $t = 1$  to the straight line  $sA + (1-s)B$  which parameterizes the line from  $B$  when  $s = 0$  to  $A$  when  $s = 1$ . This shows that with the Bezier curve  $c_1(t)$  from the order  $p_0, p_1, p_2, \dots, p_n$  and the Bezier curve  $c_2(t)$  from the order  $p_n, p_{n-1}, p_{n-2}, \dots, p_1, p_0$  are related by  $c_1(t) = c_2(1-t)$ . Alternatively, this is a property of the basis functions and their symmetry  $\beta_i^n(t) = \beta_{n-i}^n(1-t)$ .

## 14.4 Convex hull property:

A Bezier curve lies in the convex hull of the control points. To explain this property we first need to define the convex hull of a set of points. First a convex set is a set that contains the

line segment joining every pair of points in the set (see diagram below). The convex hull is the smallest convex set containing a set of points, in terms of set theory the convex hull is the intersection of all convex sets that contain the point set.

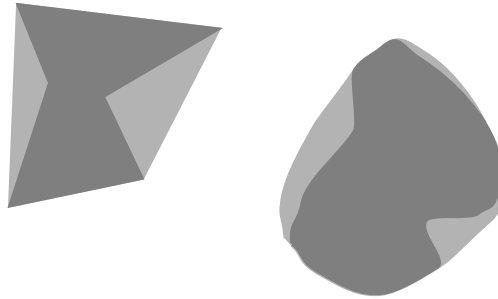


Figure 3: Convex sets and nonconvex sets

In the figure above, the darker shaded sets are not convex and the lighter shaded sets are convex.

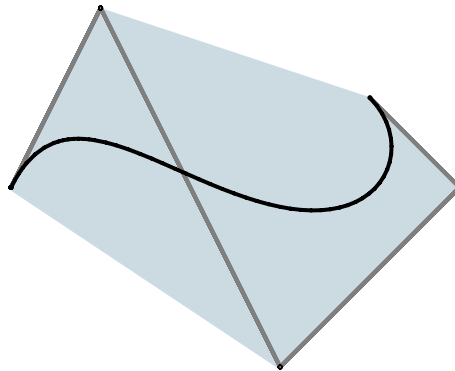


Figure 4: Convex Hull Property

That a Bezier curve lies within in the convex hull of its control points follows directly from de Casteljau's algorithm. Every point constructed in de Casteljau's algorithm lies with the convex hull of the control points, as each point lies on a line segment constructed through the control points, see diagrams below illustrating de Casteljau's algorithm an the convex hull.

### 14.5 Pseudolocal Control:

This property concerns the effect on the entire curve of changing one control point. The entire curve is affected by moving one control point. However, the major change in the curve is concentrated around one section of the curve. This is a property best seen from the basis function approach. The function

$$\beta_i^n(t) = \frac{n!}{i!(n-i)!} t^i (1-t)^{n-i}$$

has derivative equal to zero when  $t = 0$ ,  $t = 1$  and  $t = i/n$  and maximum value at  $t = i/n$ . But more importantly away from  $t = i/n$  the function is close to zero. The effect of the control point is concentrated near  $t = i/n$ , but any change affects the whole curve.

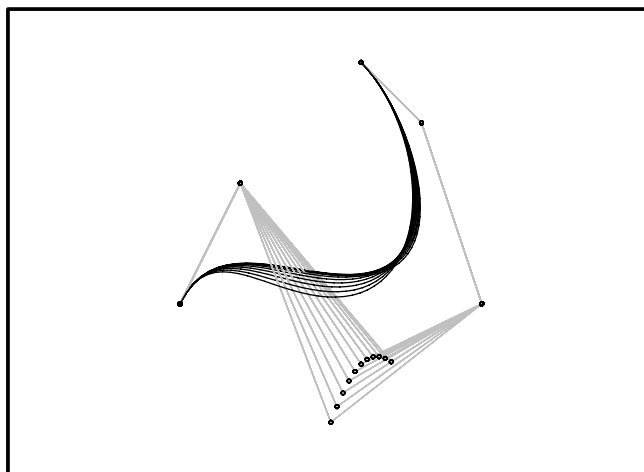


Figure 5: Pseudo-local control property

## 14.6 Variation Diminishing Property:

A planar Bezier curve intersects a line no more times than its control polyline. [Spatial bezier curves intersect a plane no more times than the control polyline, intersects the plane.] This is a very useful property when considering the possible intersection of a Bezier curve with an object. In particular, the intersection of two Bezier curves. Two Bezier curves can intersect only if their convex hulls intersect, and the intersection of a convex hull with a curve may be reduced to the intersection of a line with a curve. Variation diminishing tells us even more. It tell us how many times the curve may intersect the line.

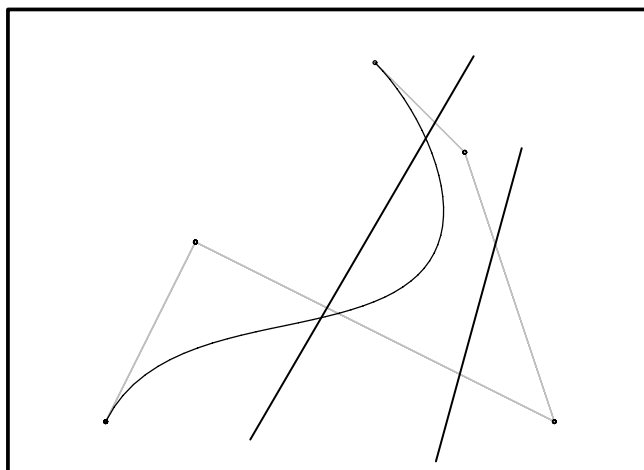


Figure 6: Variation Dimensioning Property

Proving the variation diminishing property is not exactly easy, it involves the idea of degree elevation and repeated degree elevation. Degree elevation is a simple concept. Suppose you have defined a parabola (quadratic Bezier curve) through control points  $P_0$ ,  $P_1$  and  $P_2$  to solve one problem. However, you to solve another problem, you need a cubic Bezier curve that generates the parabola. How do you generate the control points for this cubic Bezier curve?

The solution is given by degree elevation. We create a new set of control points from the control points  $p_0$ ,  $p_1$ ,  $p_2$ . [The exact manner of accomplishing this is an explored in the interactive exercises and the written exercises] In fact, the procedure involves a linear interpolation to the control polyline. It is not hard to show that linear interpolants satisfy the variation diminishing property (see the figure below). When we repeatedly applying degree elevation to a control polyline, it is true that the new control polylines approach the curve which they generate. Therefore, the original curve must satisfy the variation diminishing property.

## 14.7 Exercises

1. CONVEX HULL: Construct the convex hull of the sets below.

- (a) The set of points in the diagram below

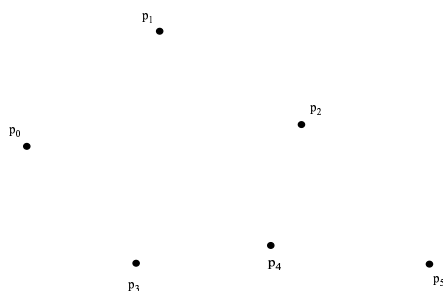


Figure 7: Construct Convex Hull of These Points

- (b) The curves below the circle, curves and the line segments in the diagram below

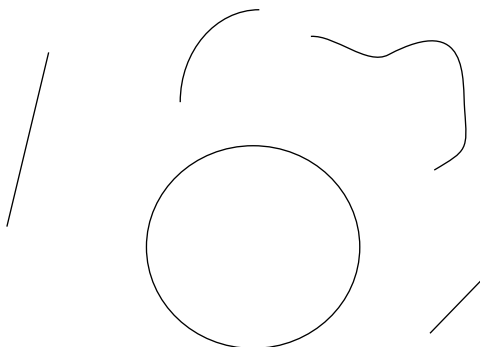


Figure 8: Construct Convex Hull

2. VARIATION DIMINISHING PROPERTY: Given control points in the diagram below.

- Determine the number of times the Bezier curve could intersect the lines  $L_1$ ,  $L_2$ , and  $L_3$  at most.
- Sketch the Bezier curve through with the given control points and determine the number of times the curve intersects the lines.

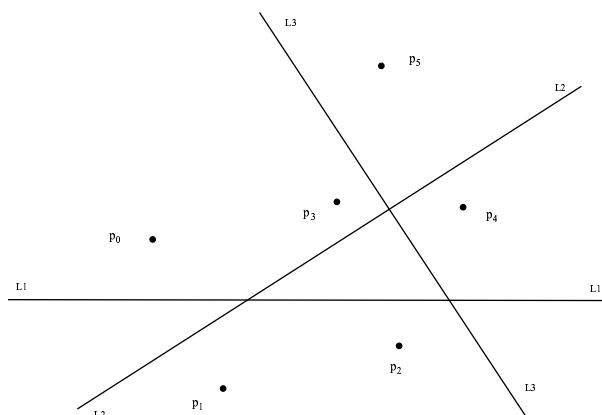


Figure 9: Variation Diminishing Property.

- DEGREE ELEVATION 1: Compute the location of the control points for the cubic by looking at the barycentric coefficients of  $P_0, P_1, P_2$  for the parabola and the barycentric coefficients for the cubic  $Q_0 = P_0, Q_1, Q_2, Q_3 = P_2$  in written as standard polynomials, that is express both in the form  $a_0 + a_1 t + a_2 t^2 + 0 t^3$ . In order for the curves to be equal the curves as polynomials must be equal, and you have equations to solve for the points. Test this solution in the applet for constructing Bezier Curves.
- DEGREE ELEVATION 2: Determine the location of the control points for a quartic curve (4th degree) to be equal to a cubic curve. First use the Bezier Curve applet to approximate the location from successfully completing the first two exercises. Then, determine the exact location of the points as in the above exercises.
- DEGREE REDUCTION: It is possible sometimes to reverse the process of degree elevation to define a lower degree curve that is equal to the given Bezier curve. This process is called degree reduction and can be directly applied only to degenerate polynomial curves. If we modify the fundamental question to “given a Bezier curve of degree  $n$  find a Bezier curve of degree  $n - 1$  that is close to the given curve, then we have a problem that is solvable. Use the Bezier curve applet to investigate a solution to this problem. Generate a cubic Bezier curve, then try to approximate the cubic curve with a quadratic Bezier curve. Then construct a quartic Bezier curve and try to approximate it with a cubic Bezier curve. How did you place the control points?
- THOUGHT PROBLEM: Given a line segment inside the convex hull of the control points of a Bezier curve. Does the Bezier curve necessarily intersect the curve. How could you determine whether it does? and the number of times for which the curve intersects the line?